Brownian Web in the Scaling Limit of Supercritical Oriented Percolation in Dimension 1+1

Anish Sarkar

Indian Statistical Institute, New Delhi

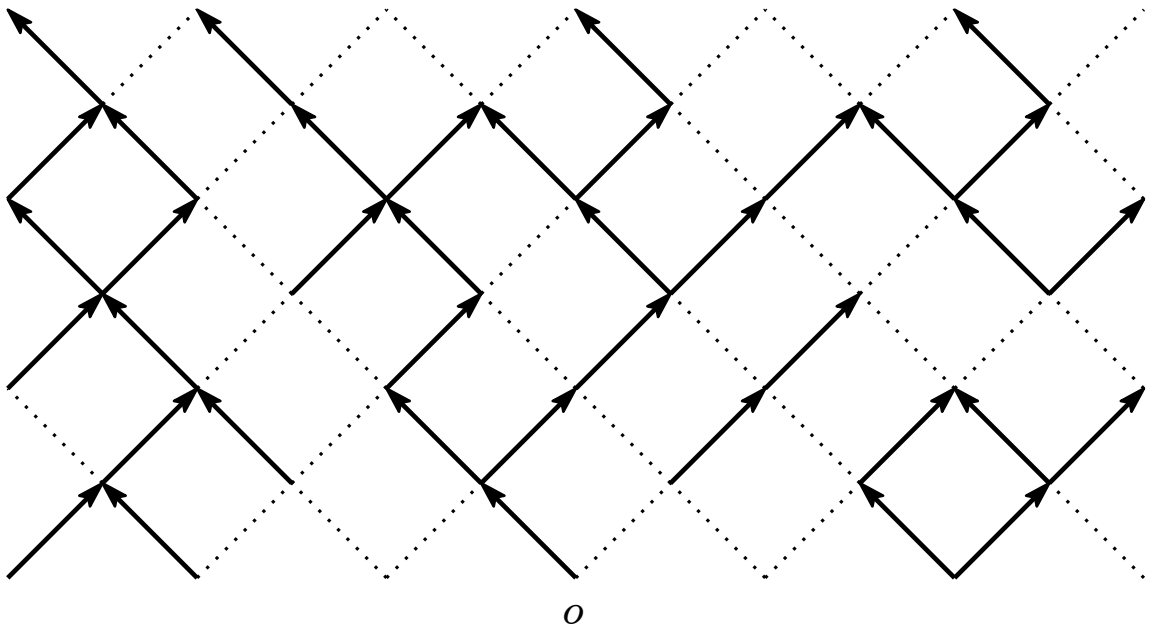
Joint work

with

Rongfeng Sun

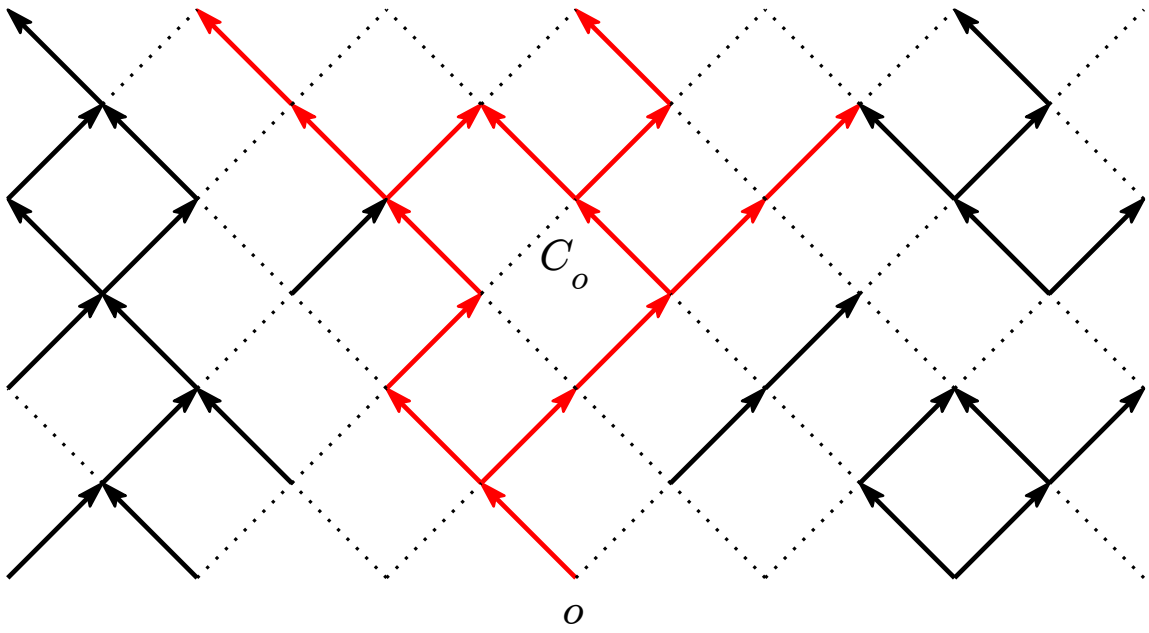
National University of Singapore

Supercritical Oriented Percolation Configuration



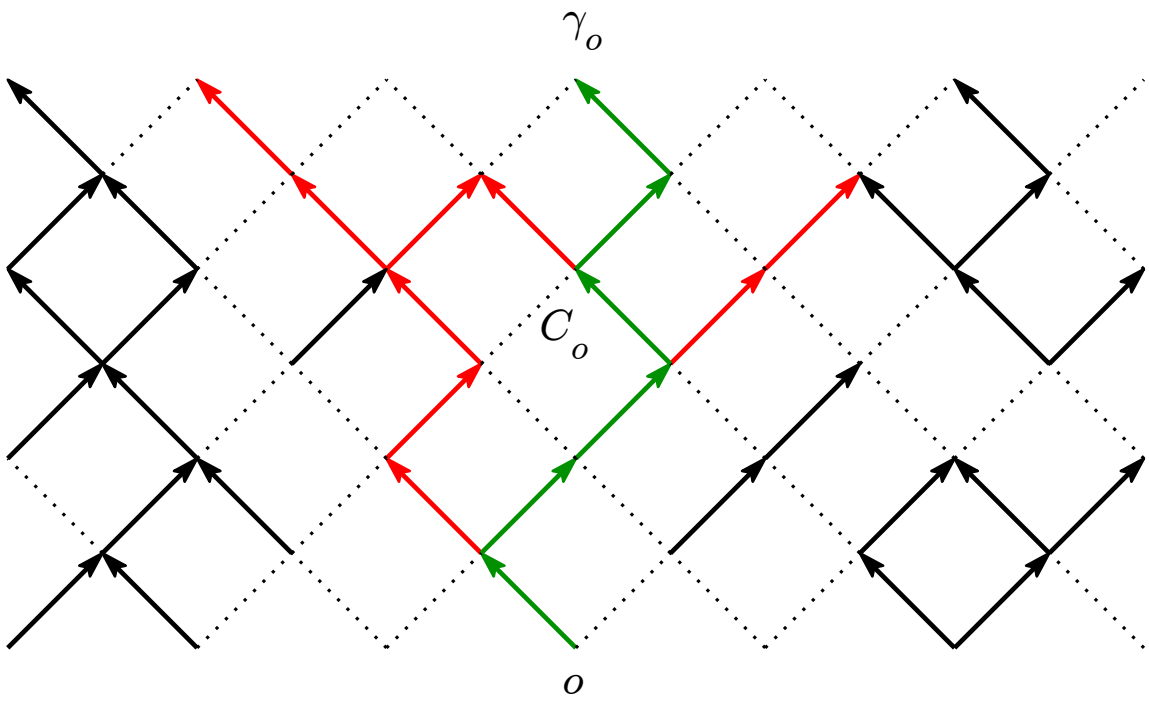
- Each directed edge in $\mathbb{Z}^2_{\text{even}} := \{(x,i) \in \mathbb{Z}^2 : x + i \text{ even}\}$ is indep. open with probability p, and closed with probability $1-p$.
- $p \in (p_c, 1)$, so that with positive probability, the open cluster C_o at the origin is infinite (o is then called a percoaltion point).
- Interested in the rightmost infinite open path γ_o starting from o, if o is a percolation point.

Supercritical Oriented Percolation Configuration



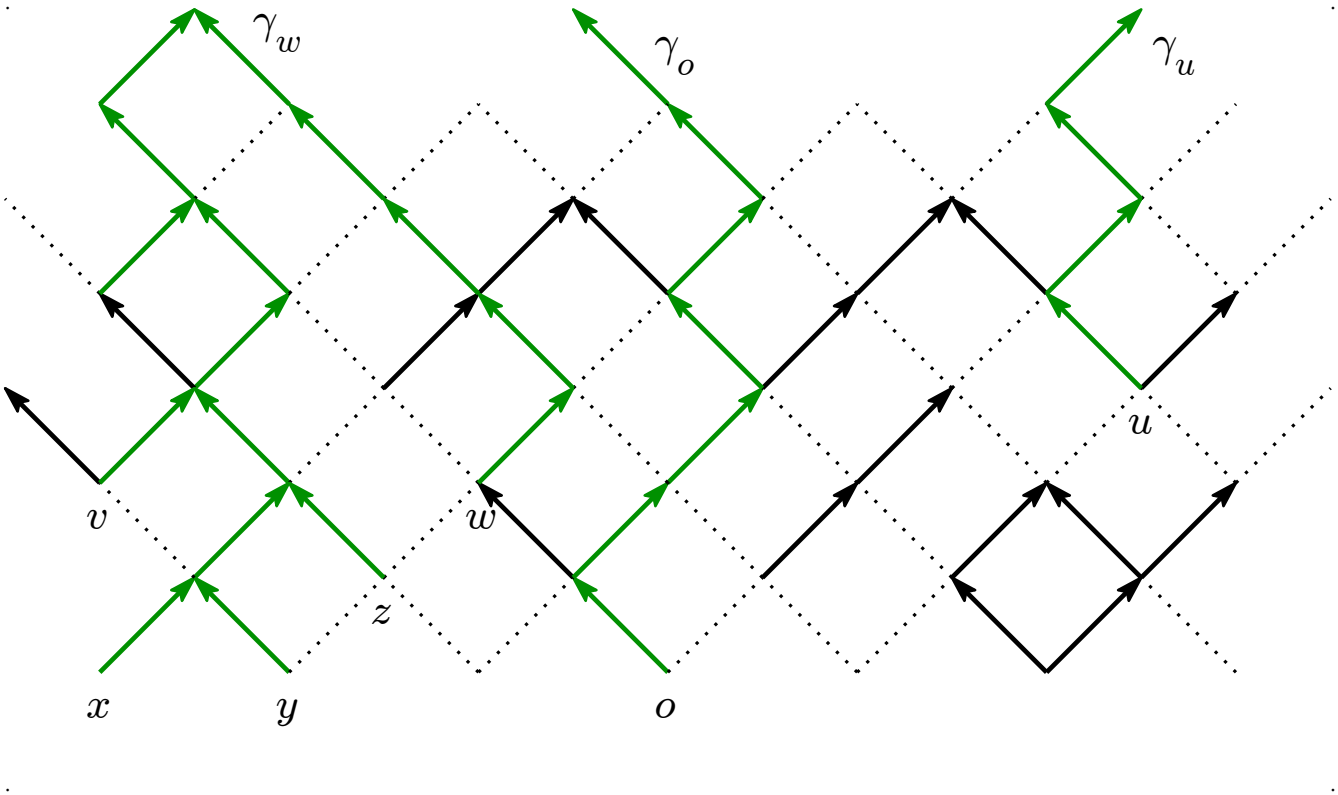
- Each directed edge in $\mathbb{Z}^2_{\text{even}} := \{(x,i) \in \mathbb{Z}^2 : x + i \text{ even}\}$ is indep. open with probability p, and closed with probability $1-p$.
- $p \in (p_c, 1)$, so that with positive probability, the open cluster C_o at the origin is infinite (o is then called a percoaltion point).
- Interested in the rightmost infinite open path γ_o starting from o, if o is a percolation point.

Supercritical Oriented Percolation Configuration



- Each directed edge in $\mathbb{Z}^2_{\text{even}} := \{(x,i) \in \mathbb{Z}^2 : x + i \text{ even}\}$ is indep. open with probability p, and closed with probability $1-p$.
- $p \in (p_c, 1)$, so that with positive probability, the open cluster C_o at the origin is infinite (o is then called a percoaltion point).
- Interested in the rightmost infinite open path γ_o starting from o, if o is a percolation point.

Collection of Rightmost Infinite Open Paths

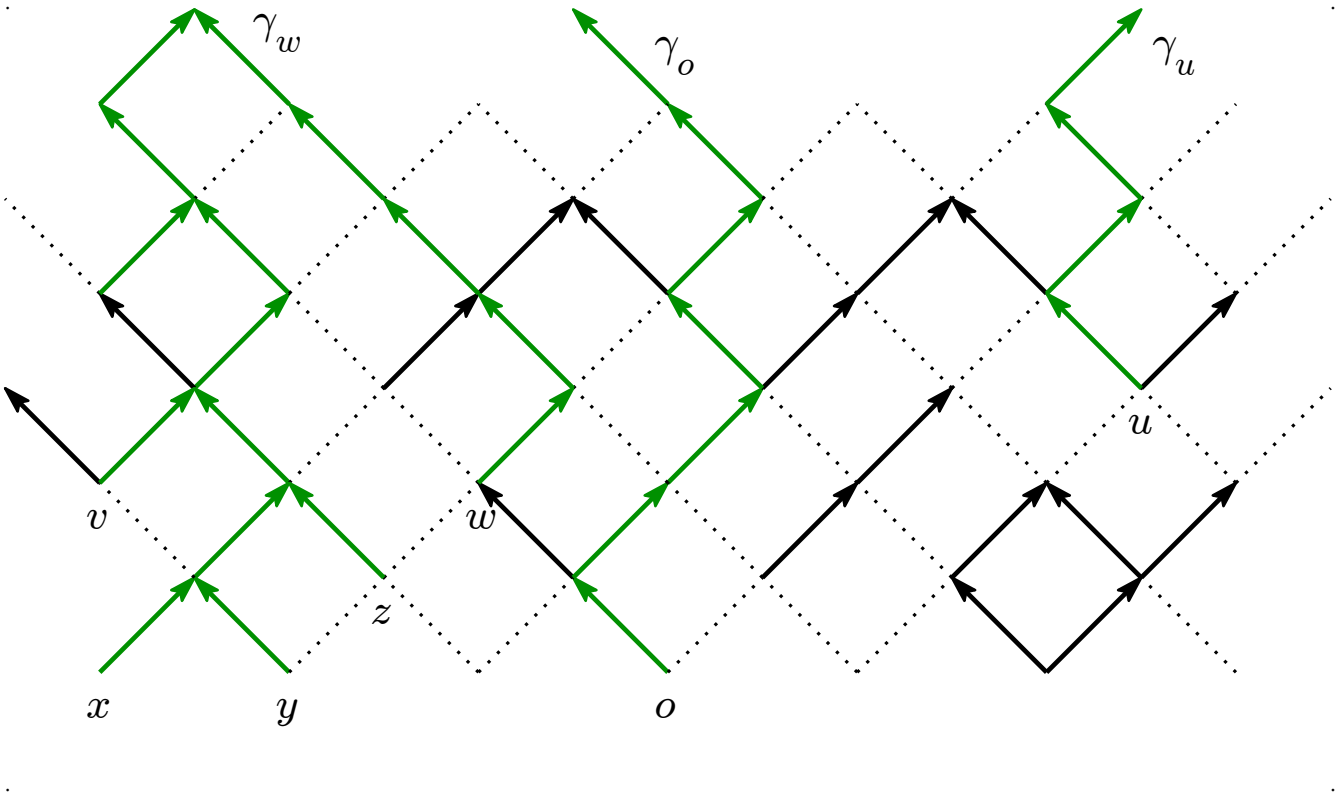


- Interested in the collection of rightmost infinite open paths,

$$\Gamma := \{\gamma_z : z \in \mathbb{Z}^2_{\text{even}}, |C_z| = \infty\}.$$

- Is there an interesting space-time scaling limit for Γ?

Collection of Rightmost Infinite Open Paths

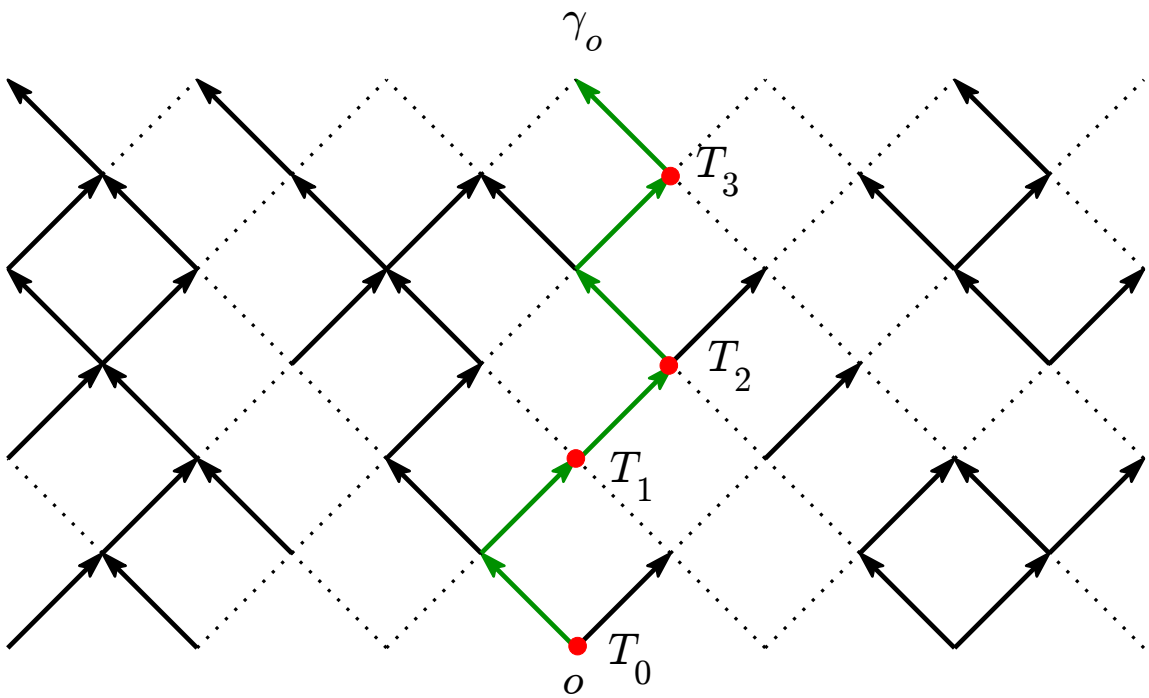


- Interested in the collection of rightmost infinite open paths,

$$\Gamma := \{\gamma_z : z \in \mathbb{Z}^2_{\text{even}}, |C_z| = \infty\}.$$

- Is there an interesting space-time scaling limit for Γ?

LLN and CLT for γ_0



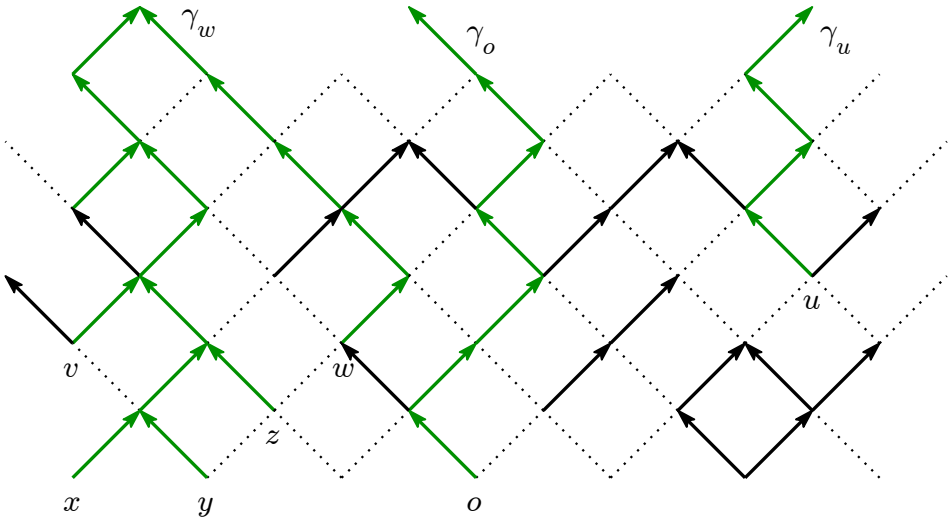
Conditional on $|C_o| = \infty$, it has been shown that

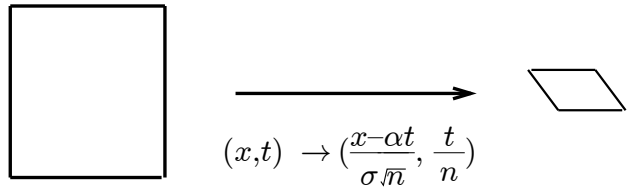
- [Durrett 84] $\exists\, \alpha(p) > 0$ s.t. $\lim_{n\to\infty} \gamma_o(n)/n = \alpha$ a.s. (subadditivity).
- [Kuczek 89] $\exists\, \sigma(p) > 0$ s.t. $(\gamma_o(n) - \alpha n)/\sigma\sqrt{n} \Rightarrow N(0,1)$. Based on break points, $(T_i, \gamma_o(T_i))$, $i \in \mathbb{N}$, with

$$\gamma_o(T_i) = \max\{x \in \mathbb{Z} : o \to (x, T_i)\}.$$

$(T_i - T_{i-1})_{i\in\mathbb{N}}$ and $(\gamma_o(T_i) - \gamma_o(T_{i-1}))_{i\in\mathbb{N}}$ are both i.i.d. with all moments finite.

Scaling Limit of Γ?





[Wu-Zhang 08]:

- Each pair of paths in Γ coalesce a.s. in finite time.
- Conjecture: After suitable centering and scaling, Γ converges to the Brownian web.

Main Result Let $p \in (p_c, 1)$. Under the shearing and diffusive scaling map $S_n : (x,t) \to (\frac{x-\alpha t}{\sigma\sqrt{n}}, \frac{t}{n})$, Γ converges in distribution to the Brownian web.

The Brownian Web $\mathcal{W}$

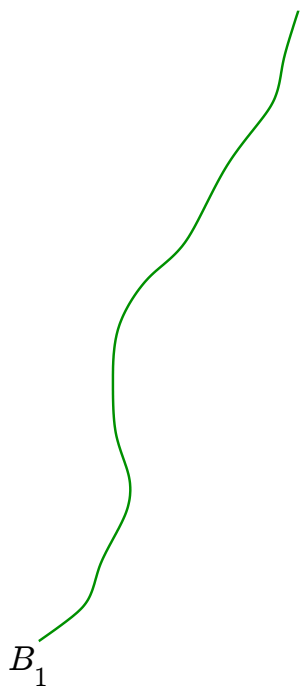
Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.

The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

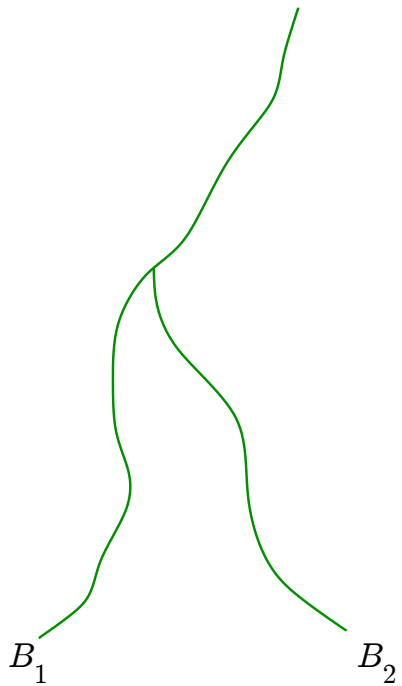
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

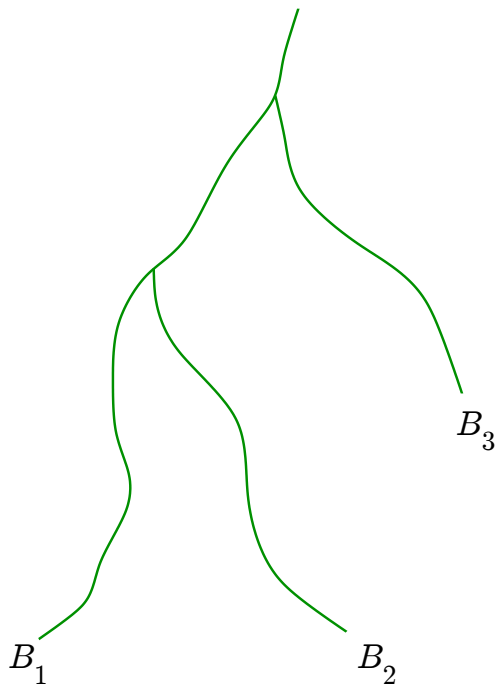
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

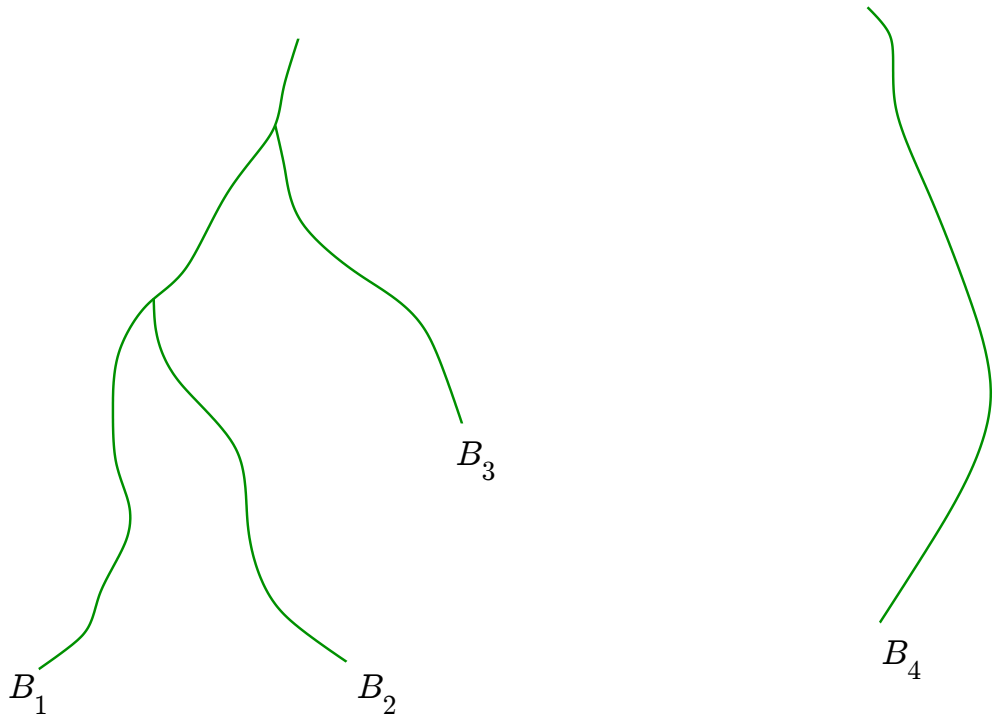
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

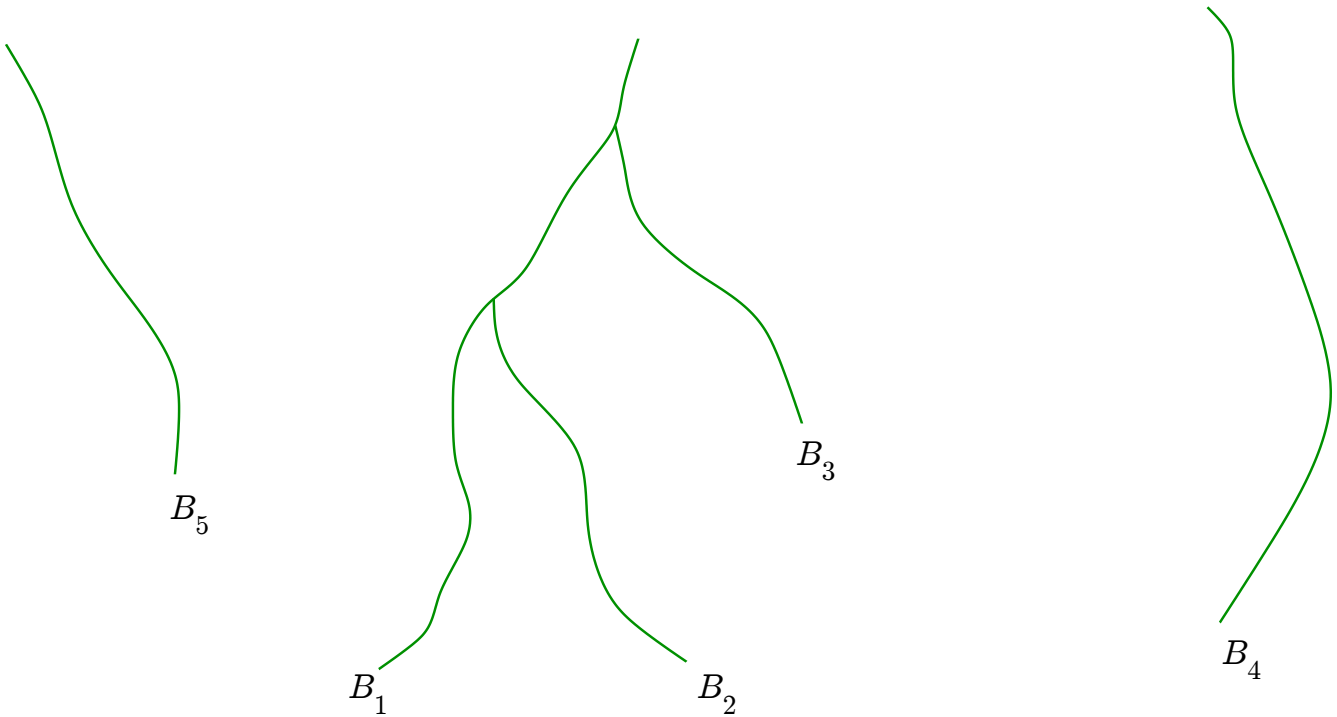
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

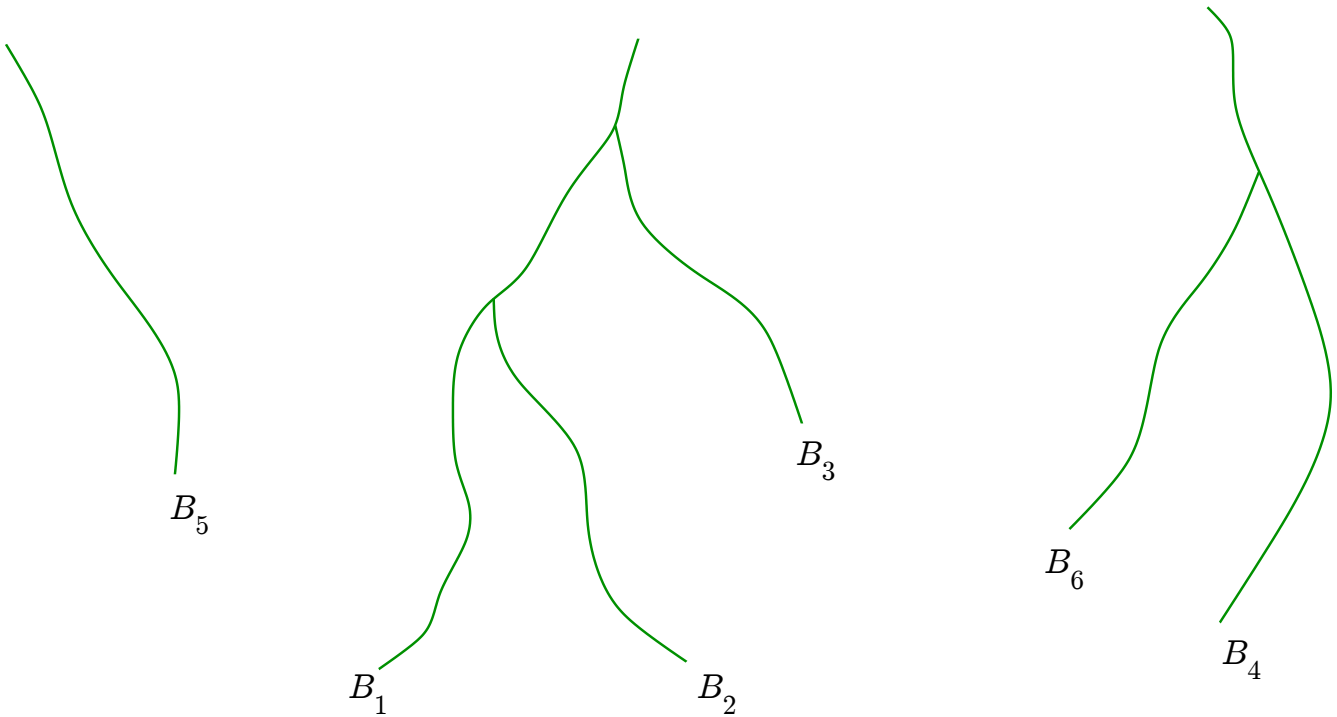
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

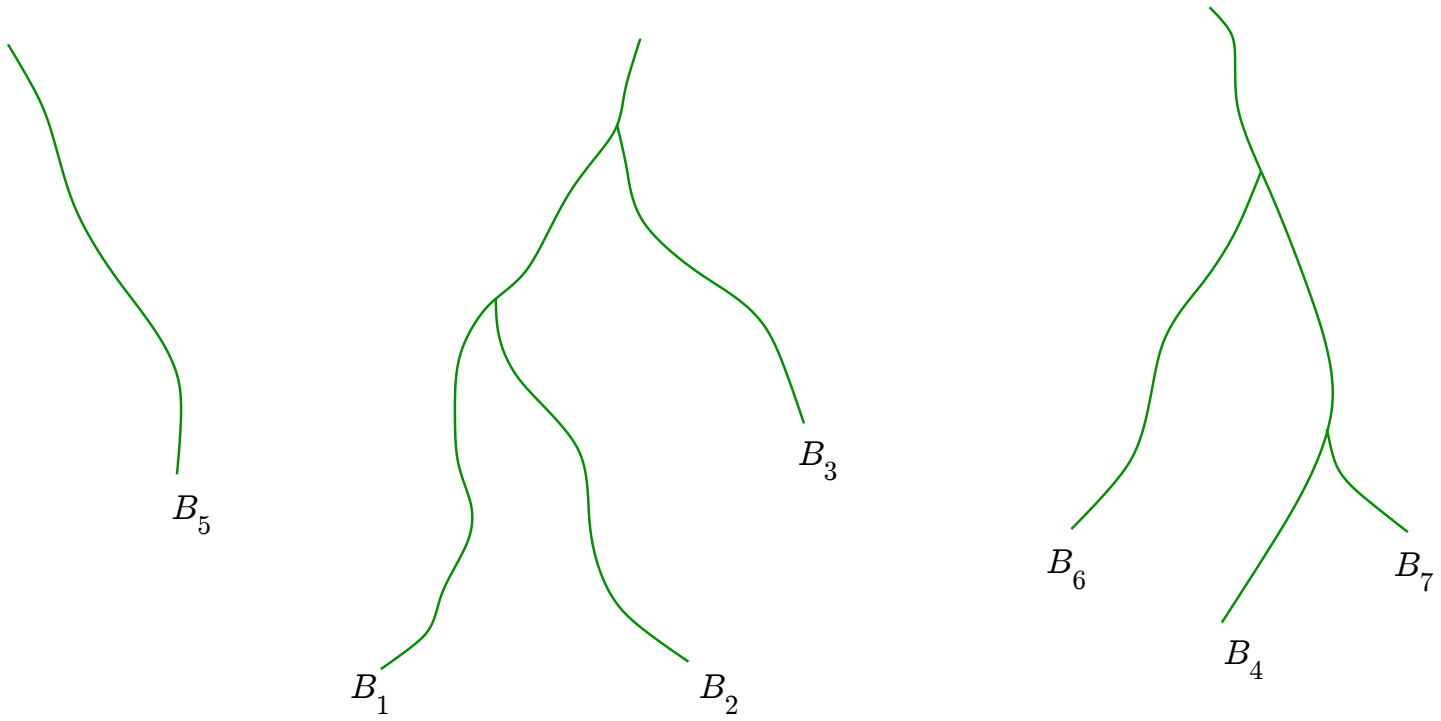
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

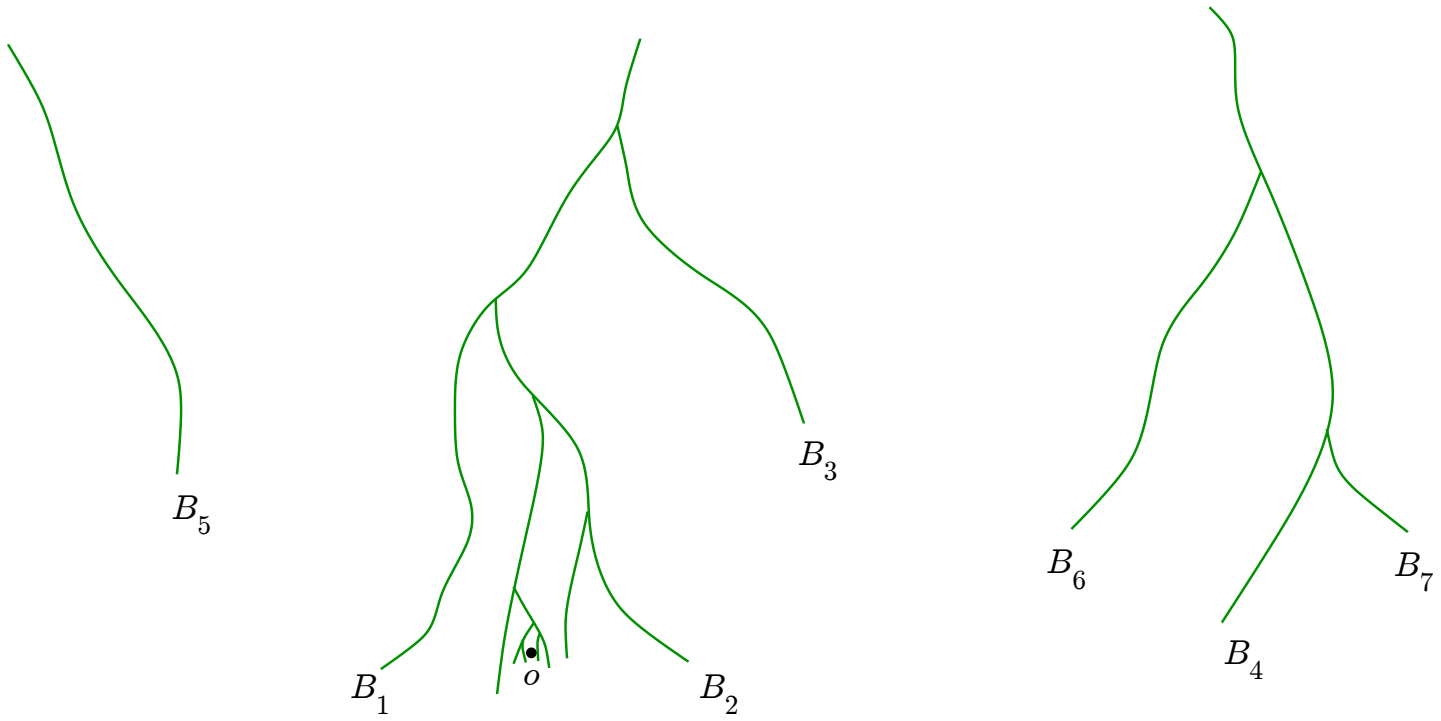
We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

Formally, the Brownian web $\mathcal{W}$ is a collection of coalescing Brownian motions, starting from every point in the space-time plane $\mathbb{R}^2$.

We can construct $\mathcal{W}$ as follows. Fix a deterministic countable dense subset $\mathcal{D} := \{z_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2$. We first construct coalescing Brownian motions starting from points in $\mathcal{D}$.



The Brownian Web $\mathcal{W}$

$\mathcal{W}$ is constructed by taking closure of $\{B_i : i \in \mathbb{N}\}$ in paths space.

Fontes-Isopi-Newman-Ravishankar Set-up.

- We treat $\mathcal{W}$ is treated as a random set of paths, with Hausdorff topology on the space of sets of paths.
- The law of $\mathcal{W}$ is independent of the choice $\mathcal{D}$ in the construction.
- For each deterministic $z \in \mathbb{R}^2$, a.s. a unique path in $\mathcal{W}$ starts from z.
- Restricting to a finite deterministic set of starting points, paths in $\mathcal{W}$ are distributed as coalescing Brownian motions.

Universality of the Brownian Web $\mathcal{W}$

The Brownian web arises the diffusive scaling limit of one-dimensional coalescing systems with sufficient space-time independence.

Systems that have been proved to converge to the Brownian web:

- Coalescing simple random walks on $\mathbb{Z}^2_{\text{even}}$ [FINR04].
- General coalescing random walks on $\mathbb{Z}^2$, with a finite $(3+\epsilon)$-th moment assumption [Newman-Ravishankar-Sun 05, Belhaouari-Mountford-Sun-Valle 06].
- Two-dimensional Poisson trees [Ferrari-Landim-Thorisson 04, Ferrari-Fontes-Wu 05].
- A two-dimensional drainage model [Gangopadhyay-Roy-Sarkar 04, Coletti-Fontes-Dias 09].
- Hastings-Levitov planar aggregation model [Norris-Turner 08].

Approximating γ_0 by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

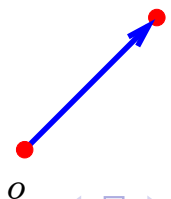
This leads naturally to the notion of an exploration cluster.

Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

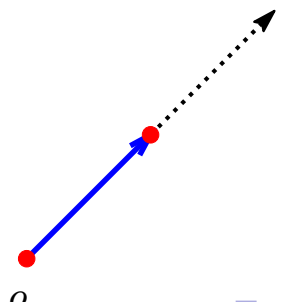


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

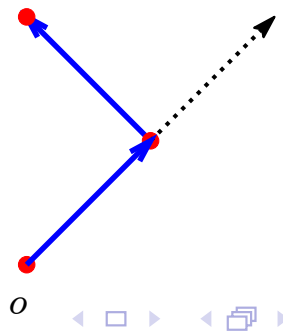


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

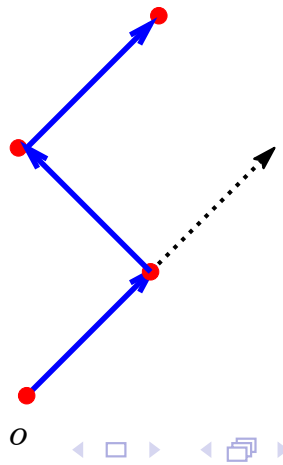


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

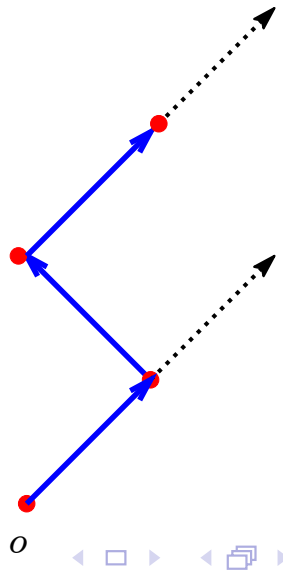


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

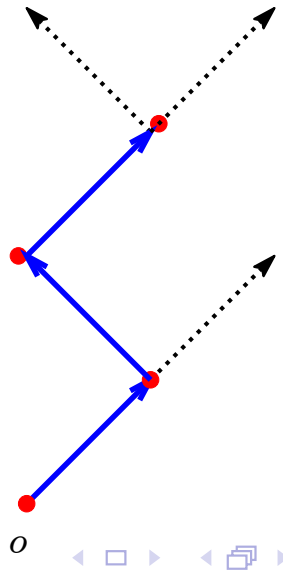


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

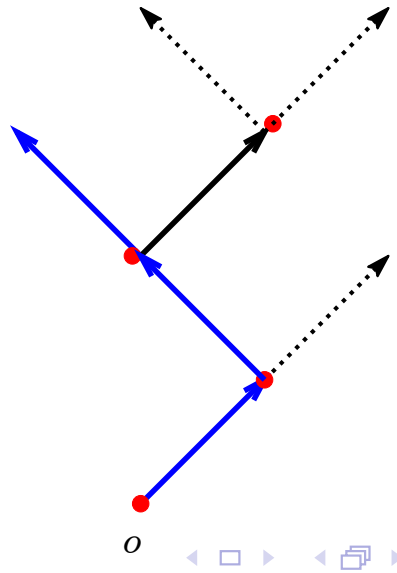


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

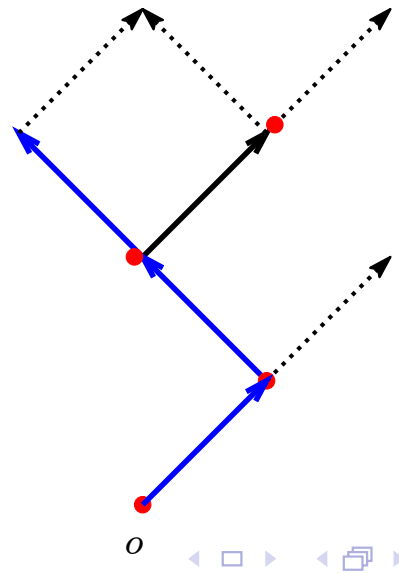


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

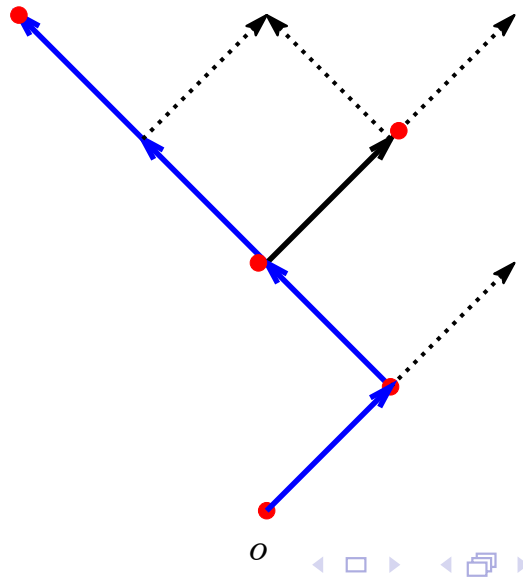


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

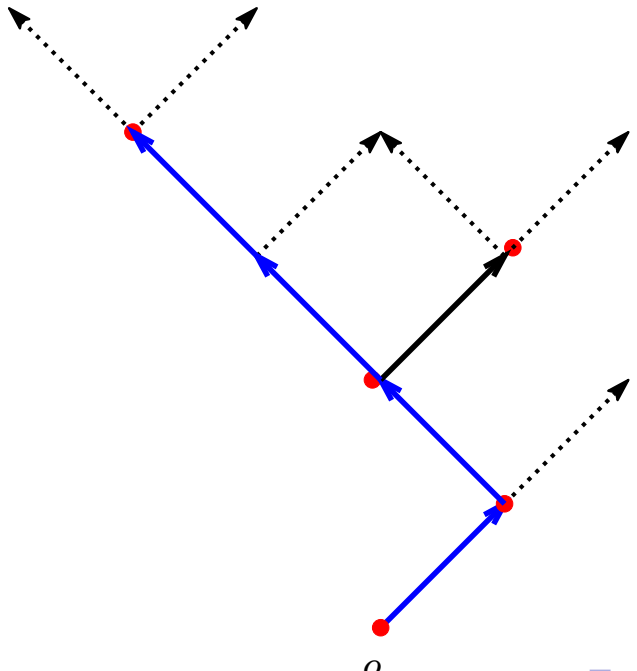


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

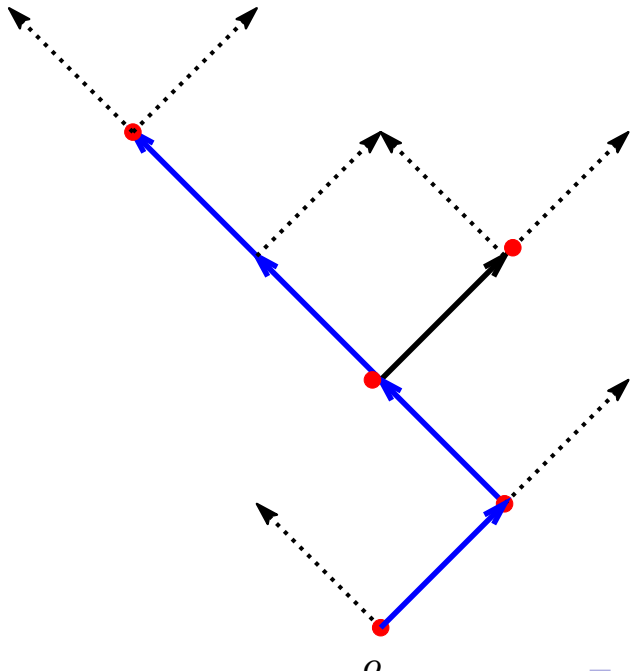


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

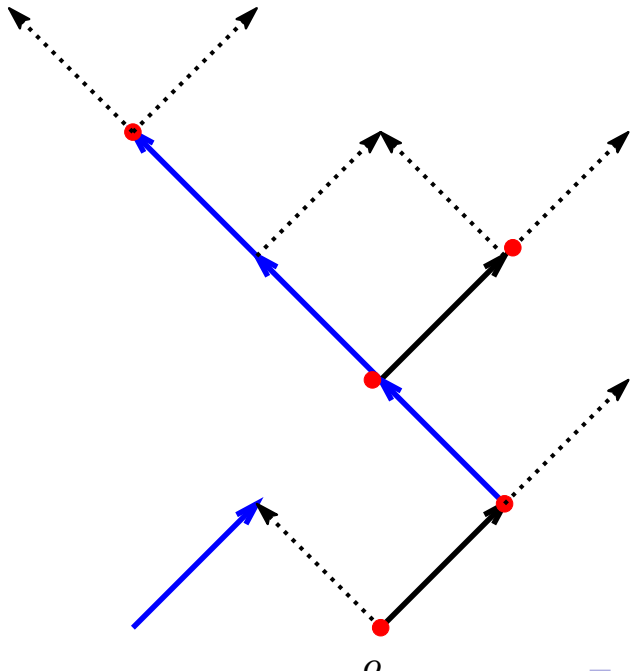


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

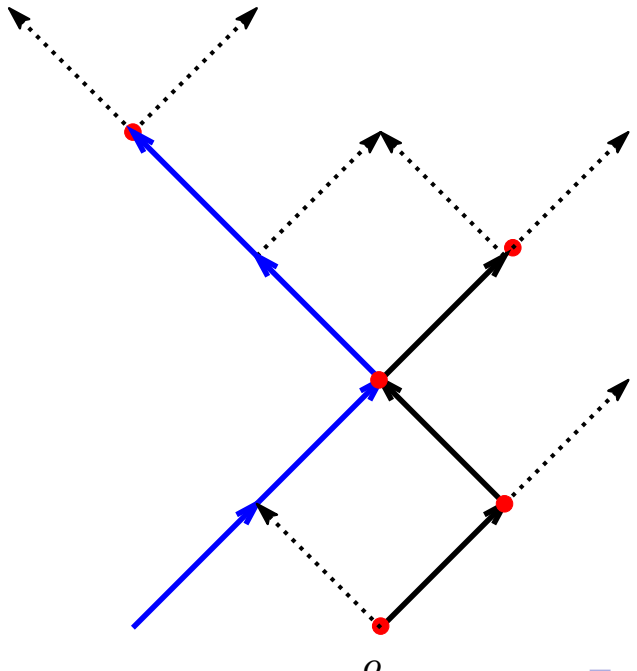


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

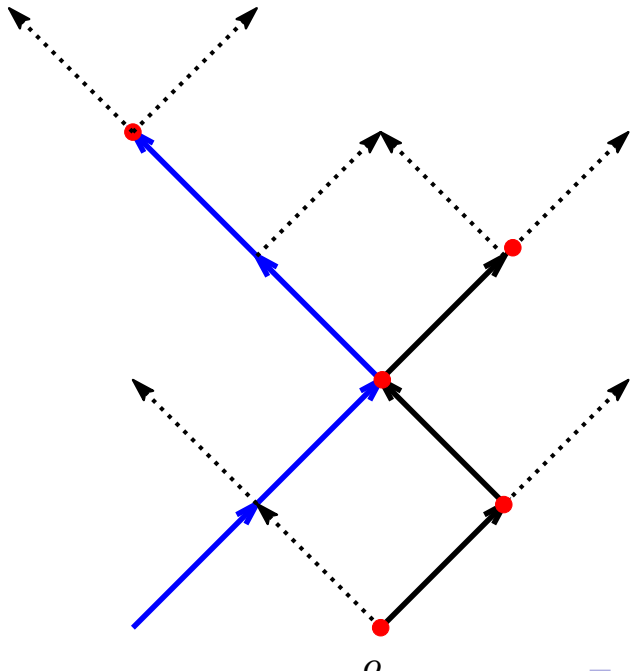


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

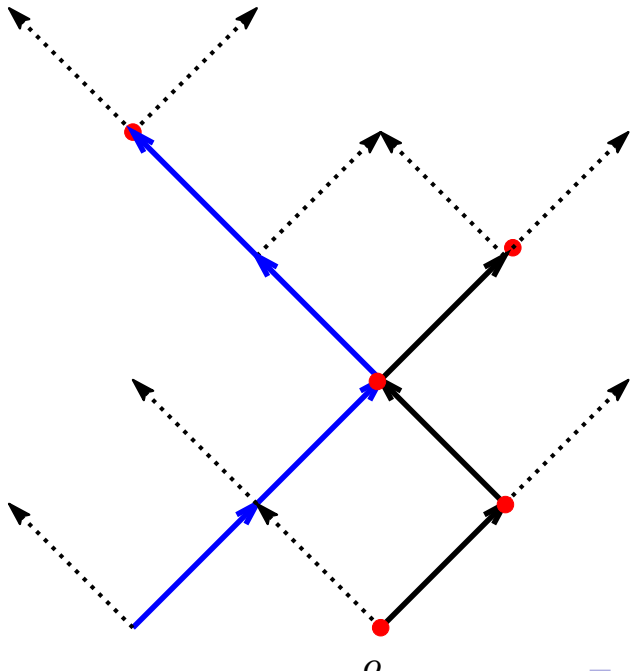


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

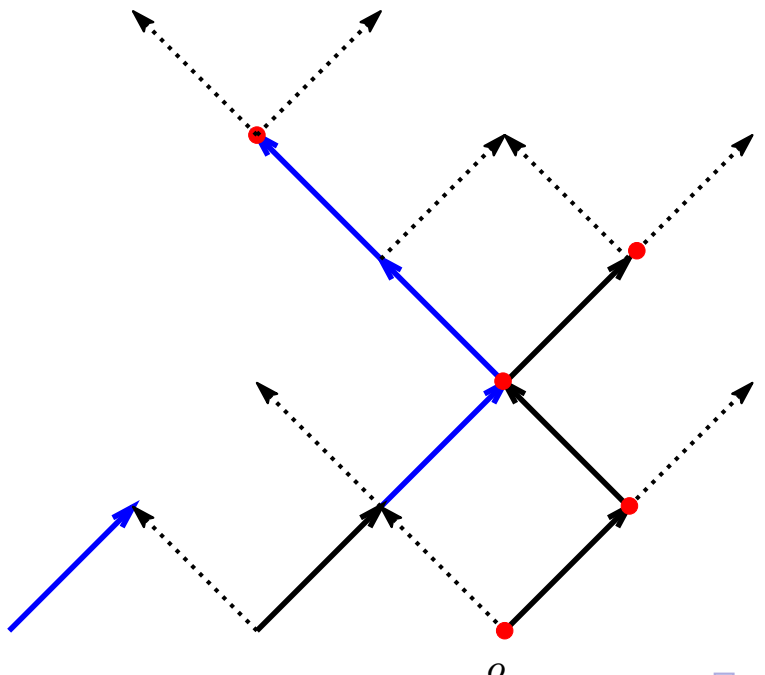


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

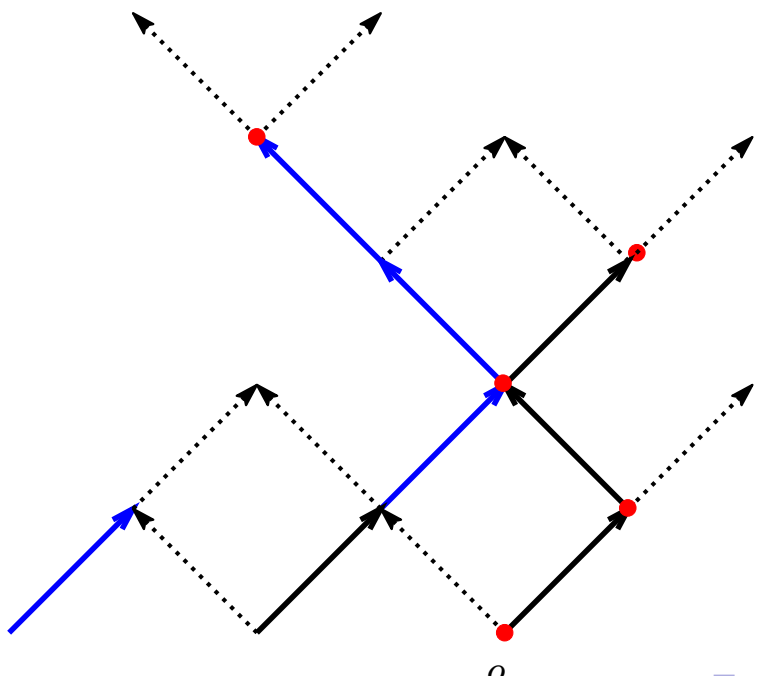


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

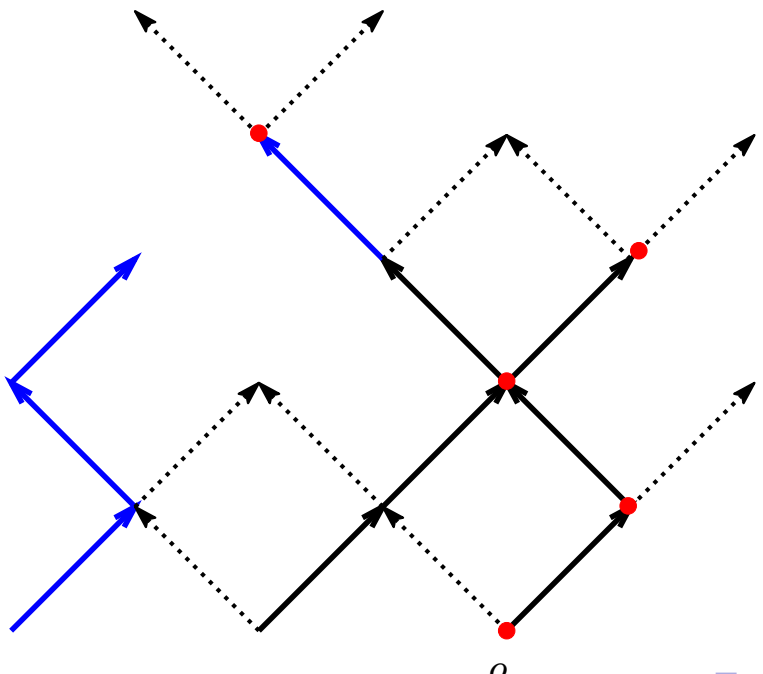


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

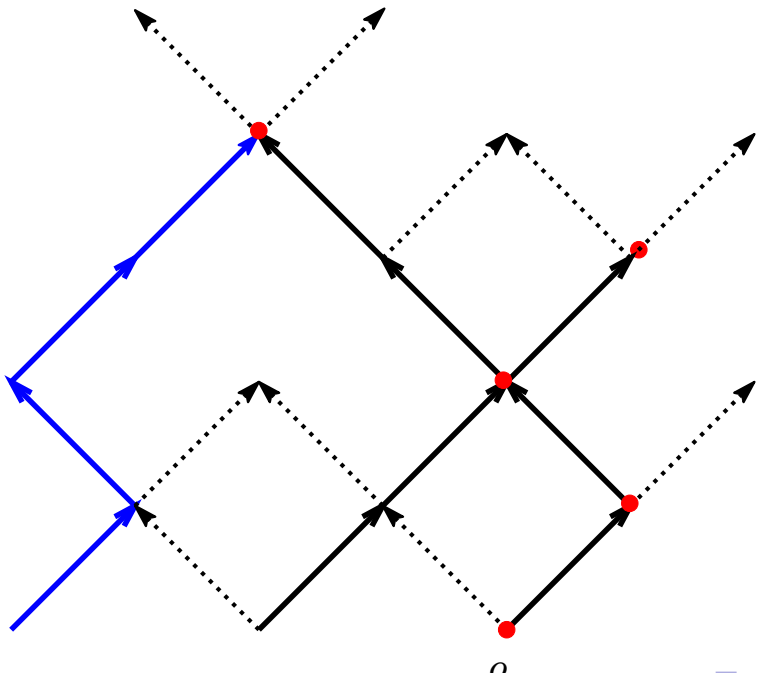


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

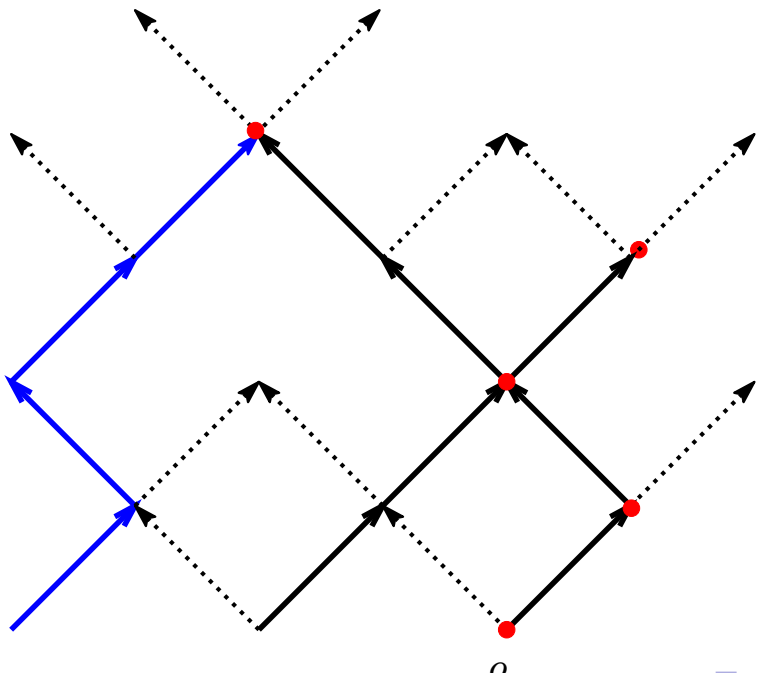


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

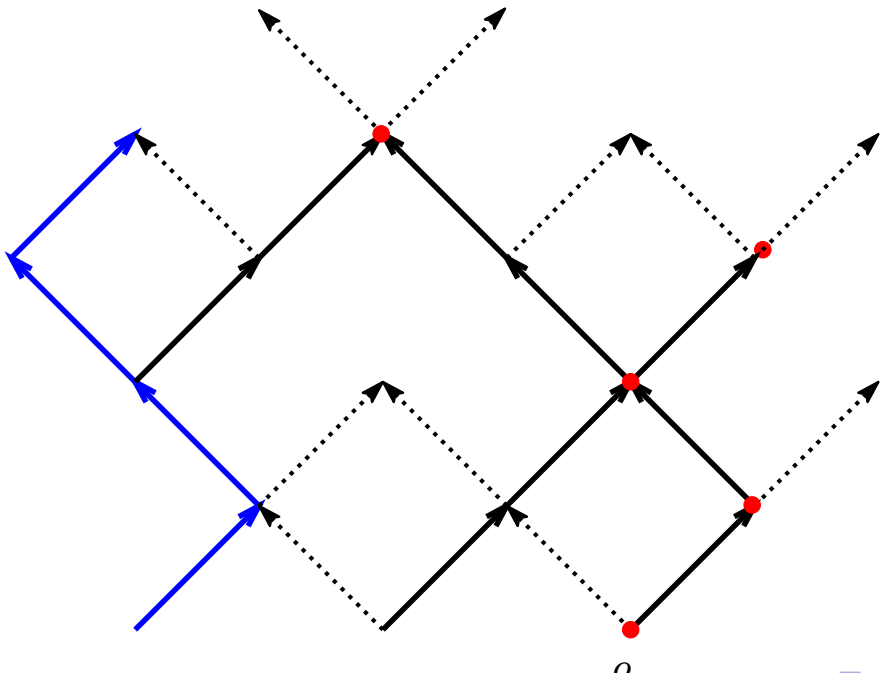


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

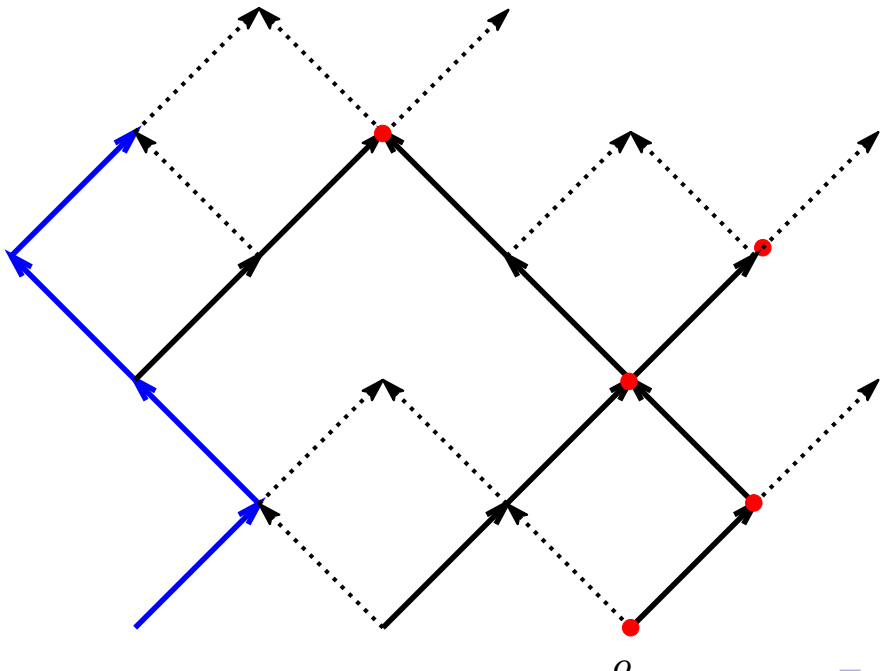


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

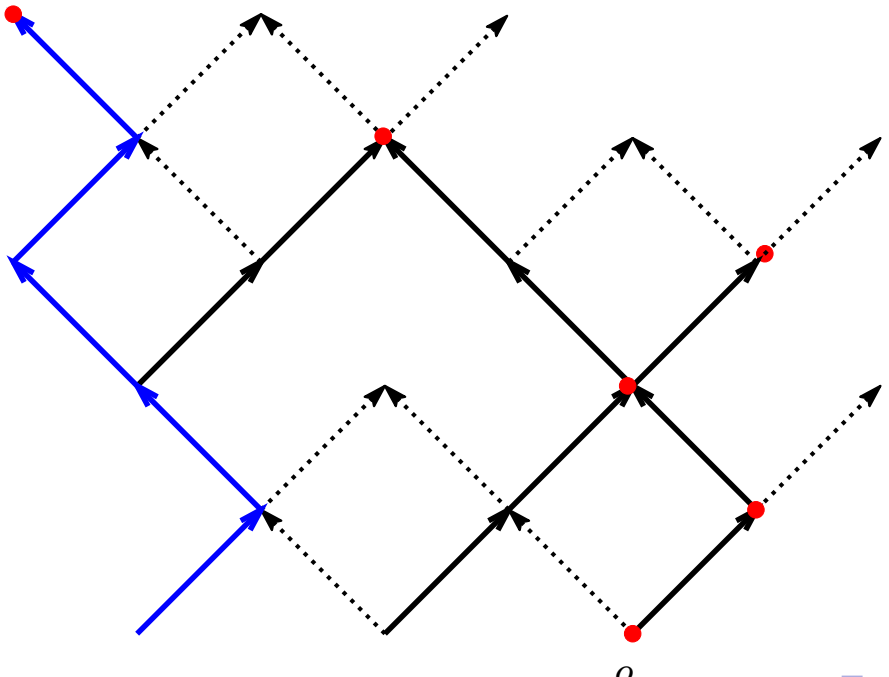


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

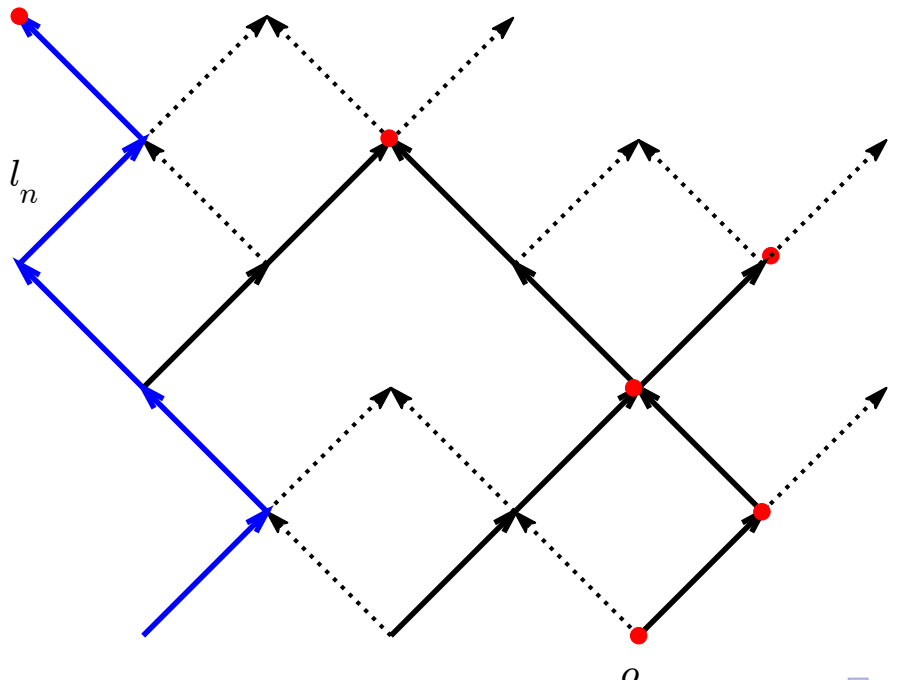


Approximation by an Exploration Cluster

Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

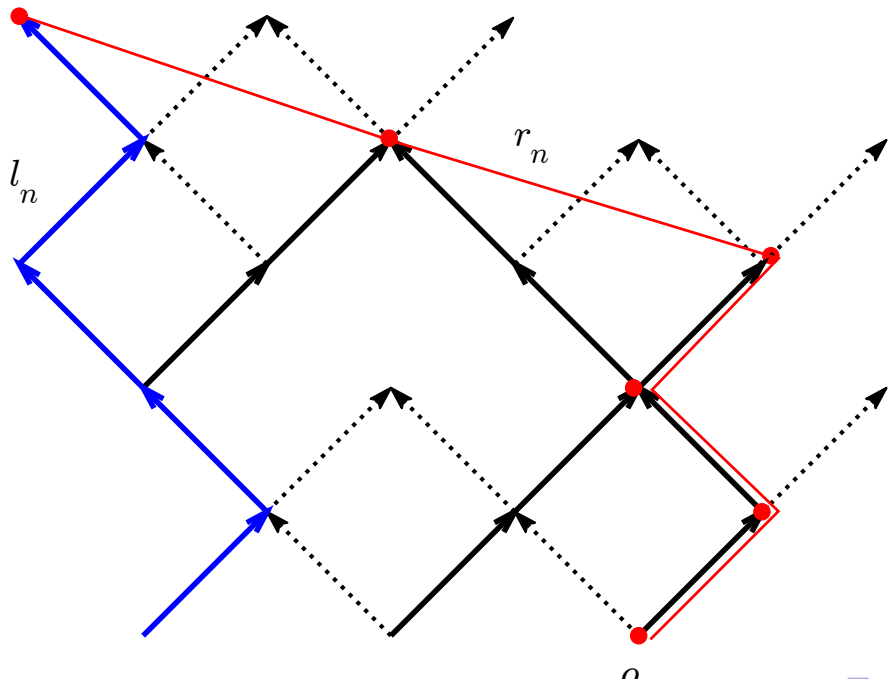


Approximation by an Exploration Cluster

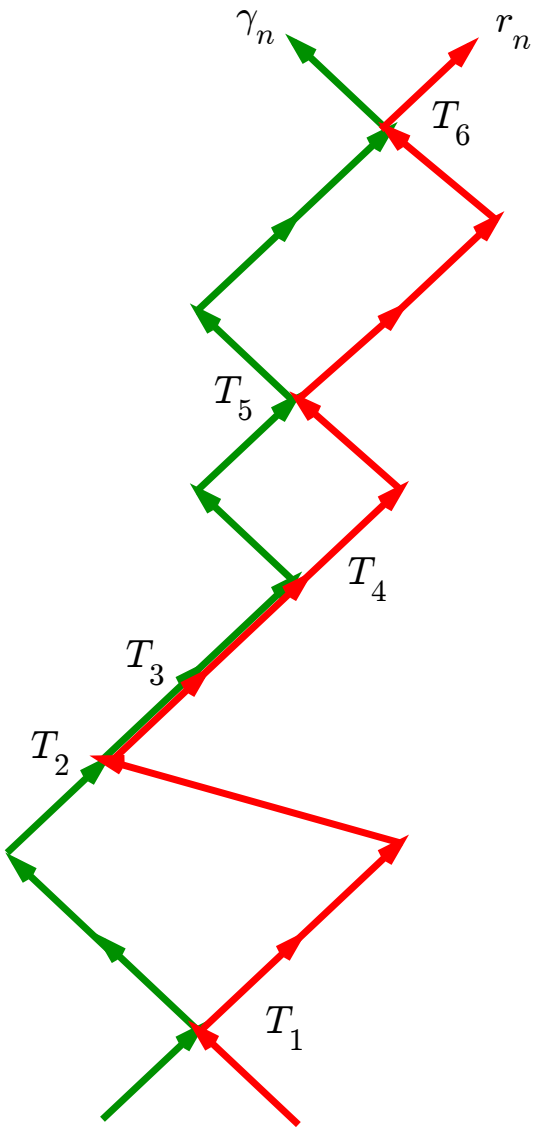
Recall that γ_o is the rightmost infinite open path if $|C_o| = \infty$.

- To avoid conditioning on $|C_o| = \infty$ in order to define γ_o, if $|C_o| < \infty$, we define γ_o to be the rightmost infinite open path starting from $(-\infty, 0]$ at time 0.
- To remove dependence on the future, for each $n \in \mathbb{N}$, we approximate γ_o by the rightmost open path starting from $(-\infty, 0]$ at time 0 and reaching time n.

This leads naturally to the notion of an exploration cluster.

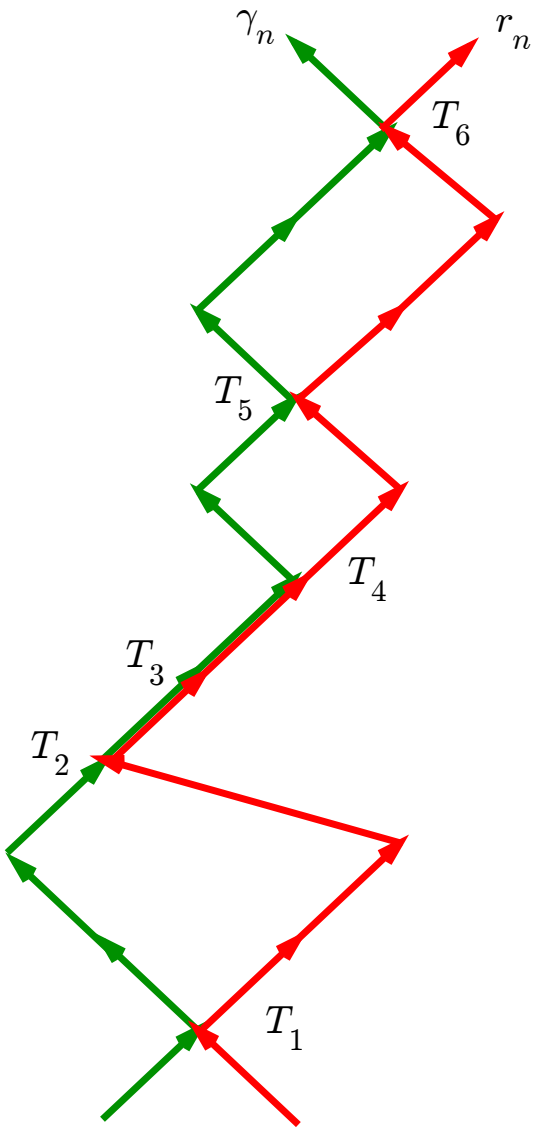


Invariance Principle for an Exploration Cluster



- Exploration cluster is bounded between γ_n and r_n for all $n \geq 0$.

Invariance Principle for an Exploration Cluster

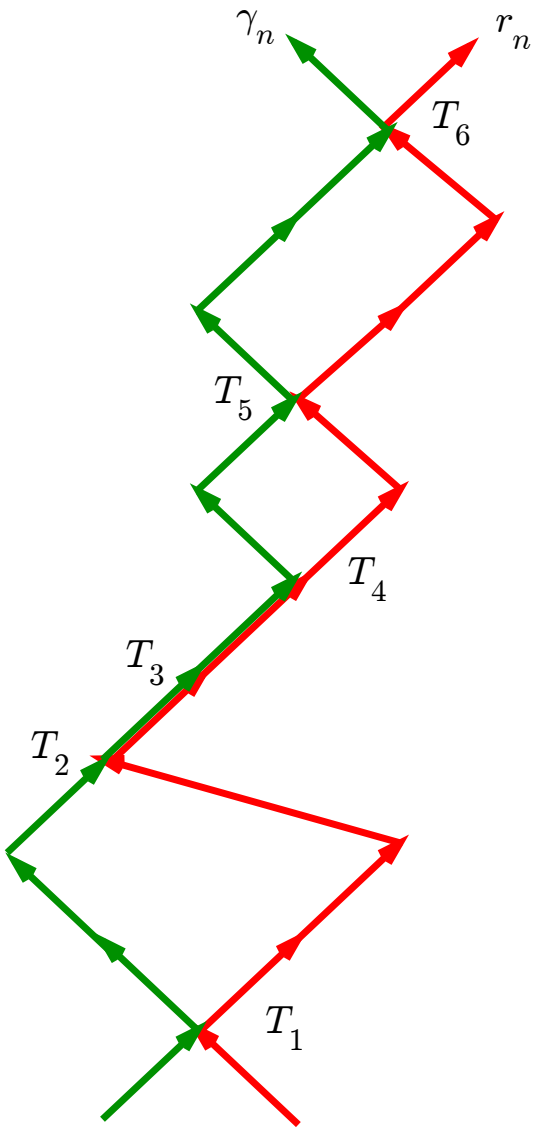


- Exploration cluster is bounded between γ_n and r_n for all $n \geq 0$.

- γ_n and r_n coincide at break points at times T_i, $i \geq 1$.

  T_i–T_{i-1}, $i \geq 2$, are i.i.d. with all moments finite.

Invariance Principle for an Exploration Cluster



- Exploration cluster is bounded between γ_n and r_n for all $n \geq 0$.

- γ_n and r_n coincide at break points at times T_i, $i \geq 1$.

  T_i–T_{i-1}, $i \geq 2$, are i.i.d. with all moments finite.

- Under the shearing and diffusive scaling map

$$S_n\colon (x,t) \to \left(\frac{x-\alpha t}{\sigma\sqrt{n}}, \frac{t}{n}\right),$$

  γ_n and r_n converge to the same Brownian motion $(B_t)_{t \geq 0}$.

When Two Exploration Clusters Intersect

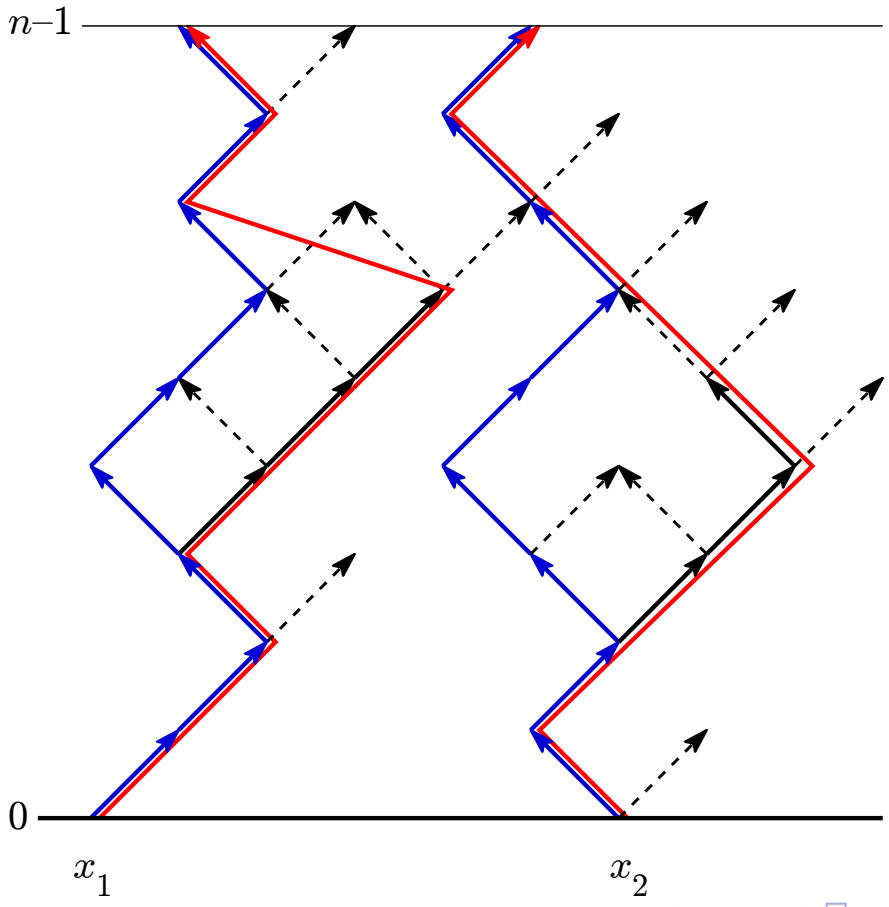
Different exploration clusters evolve independently before intersecting.

If two exploration clusters start at the same time:

When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

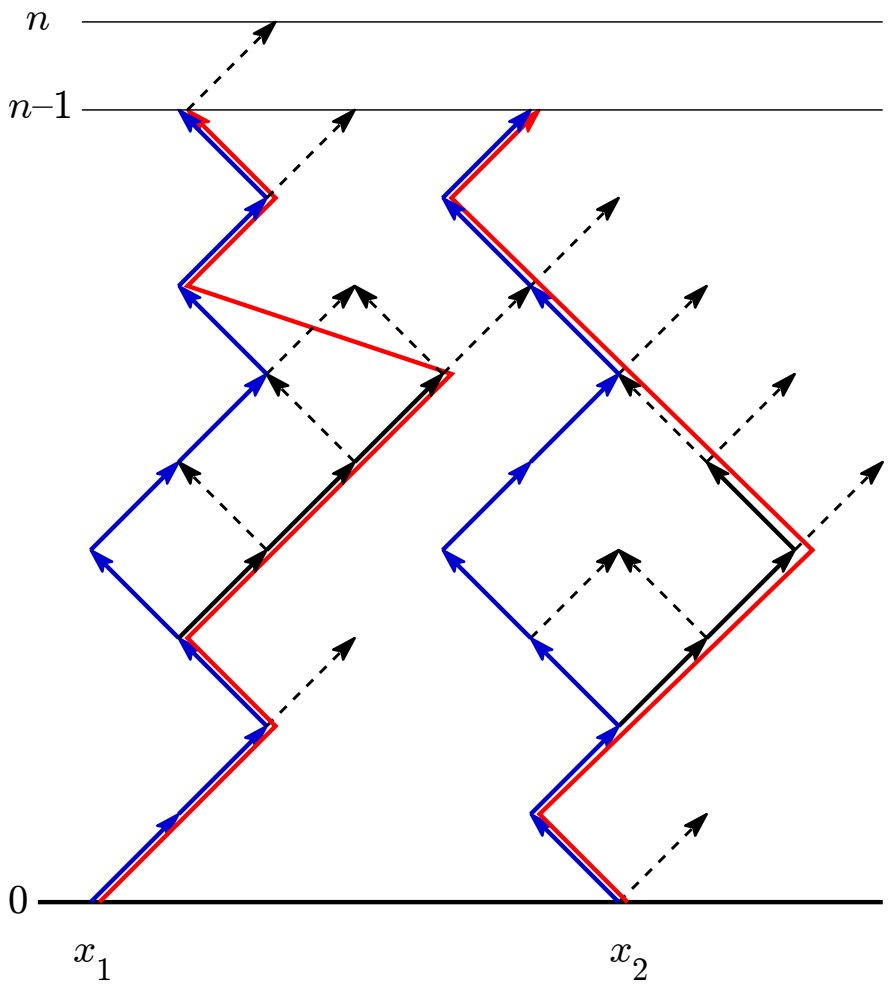
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

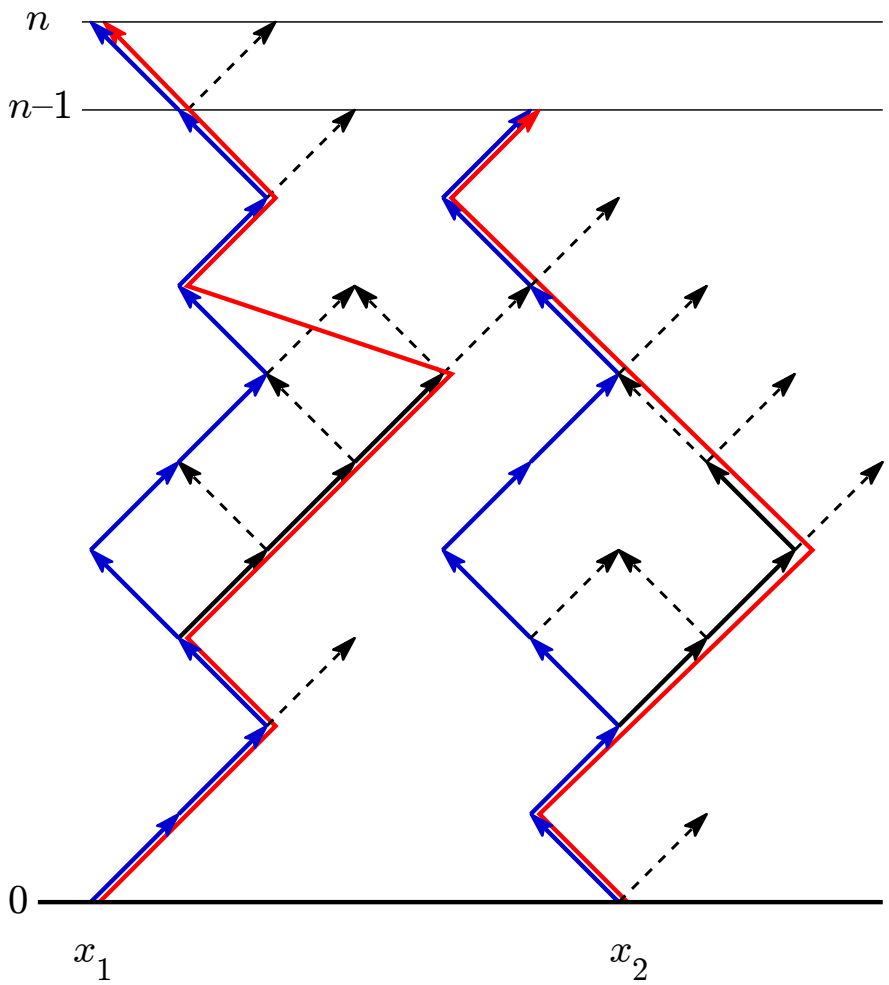
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

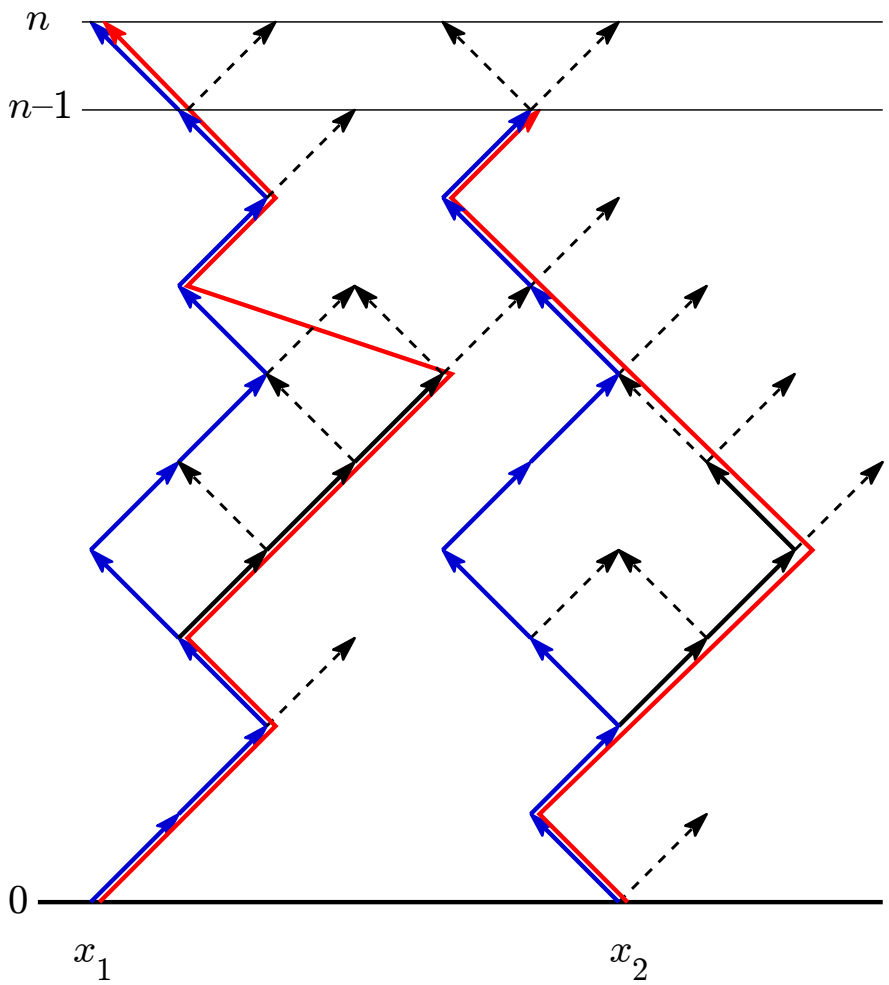
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

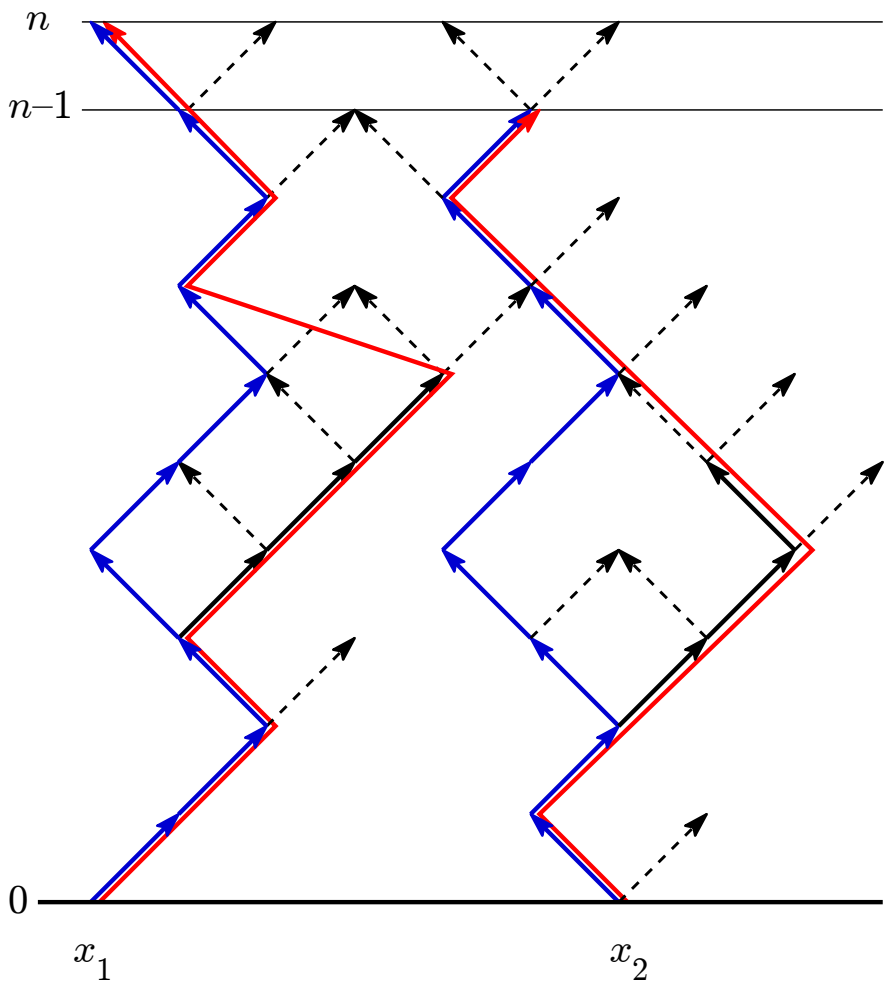
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

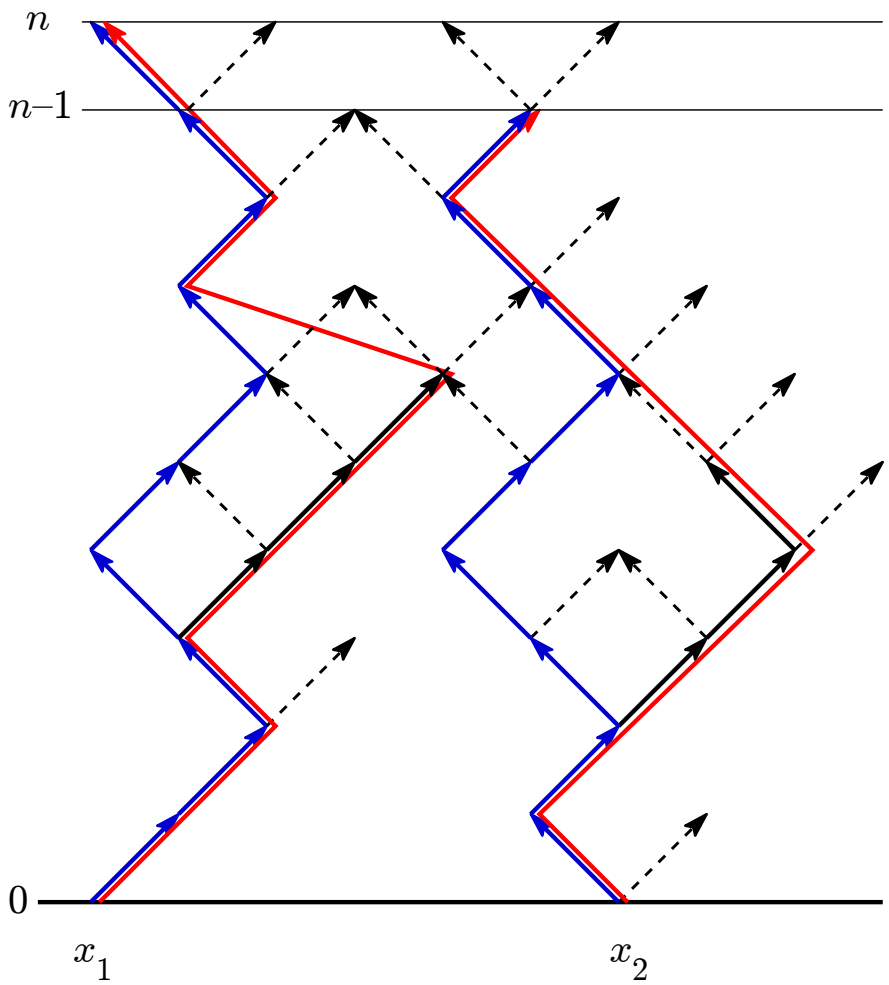
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

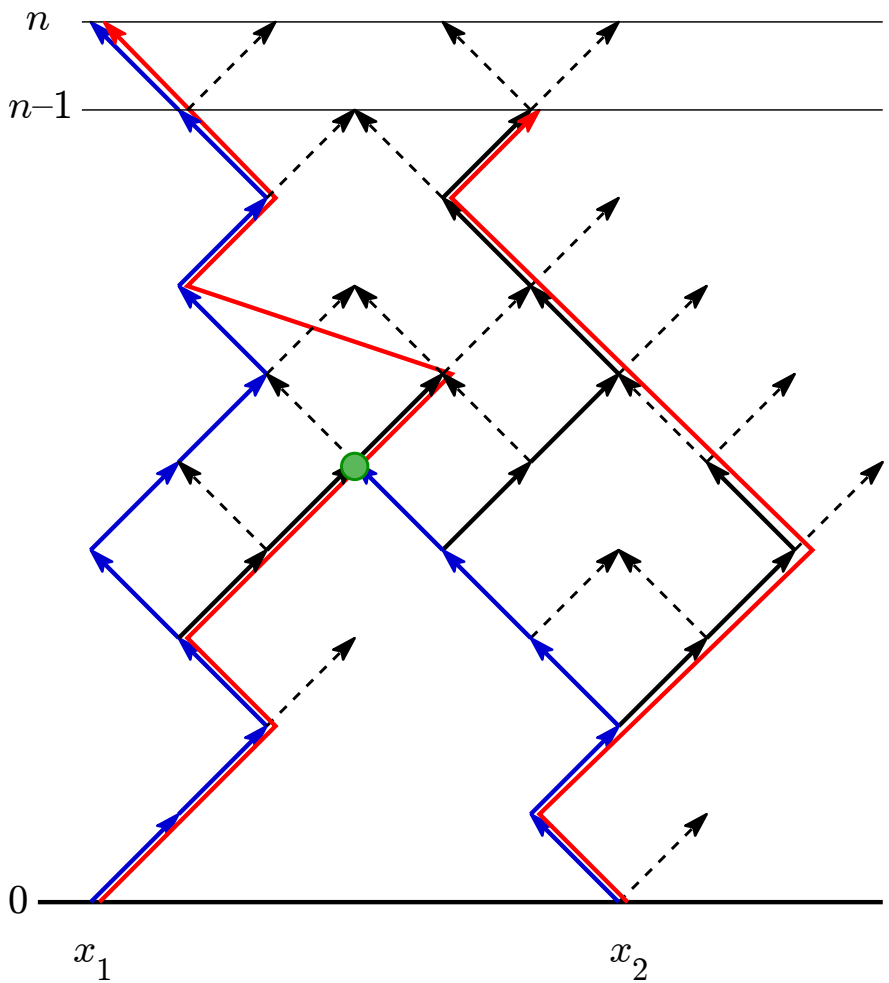
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

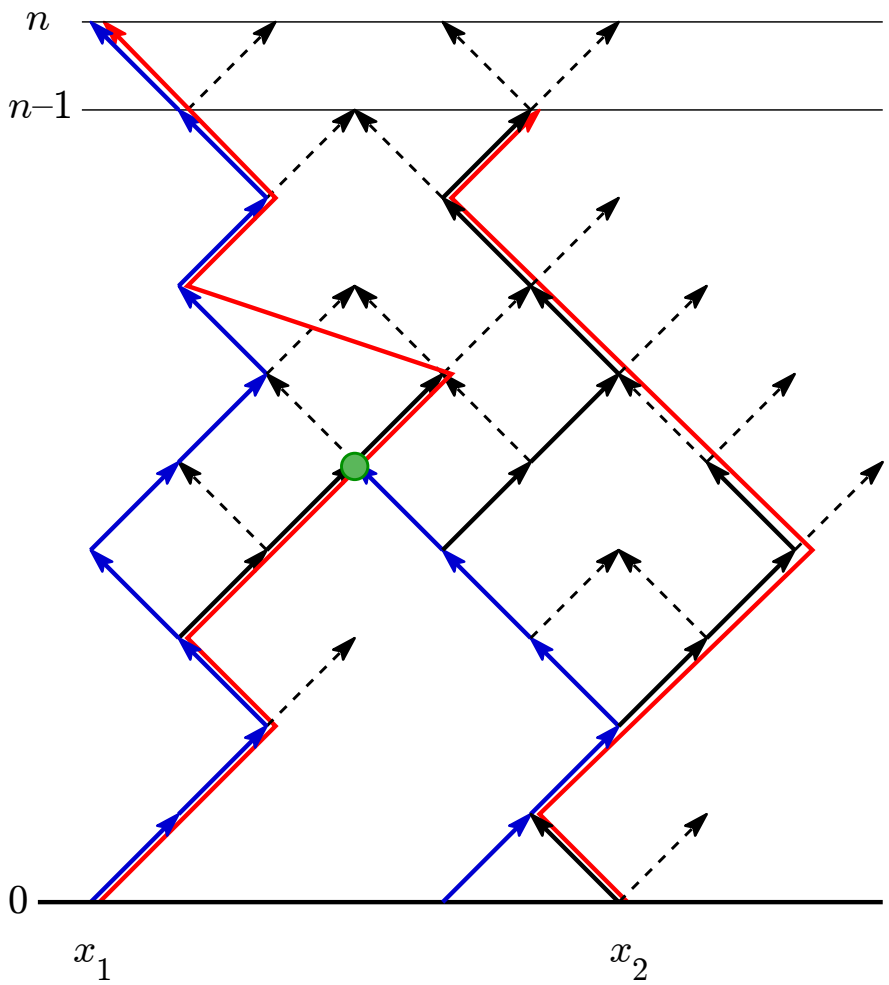
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

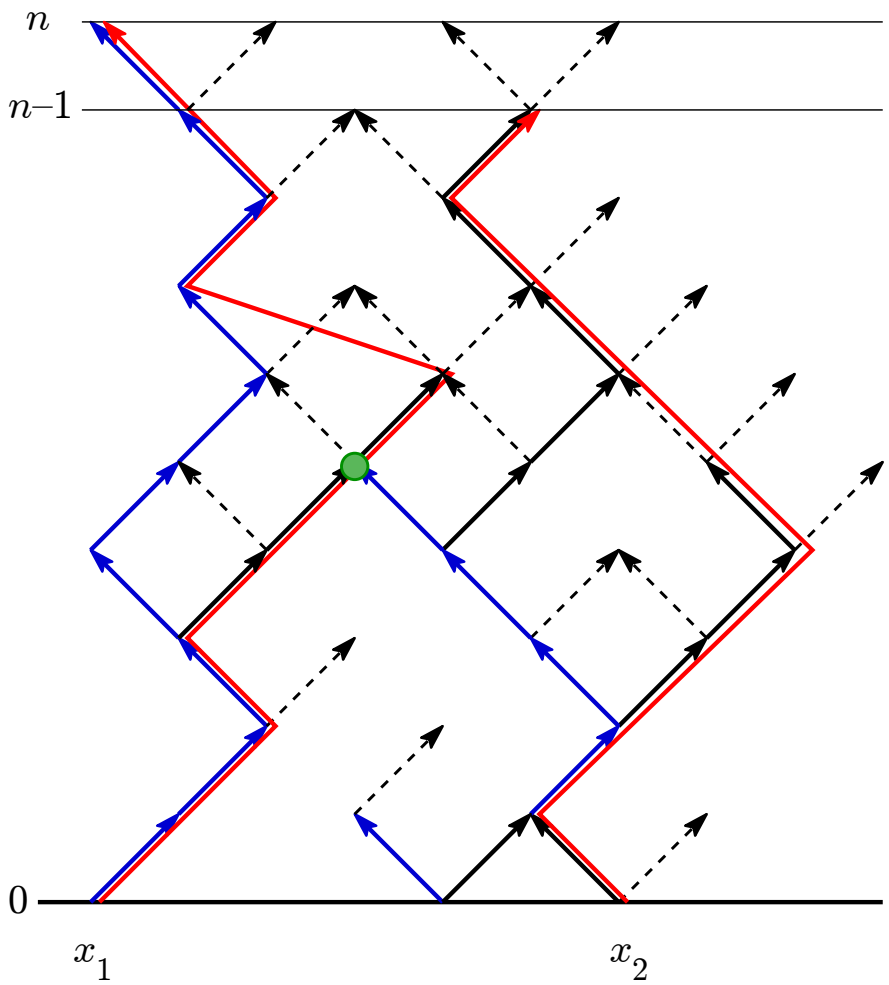
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

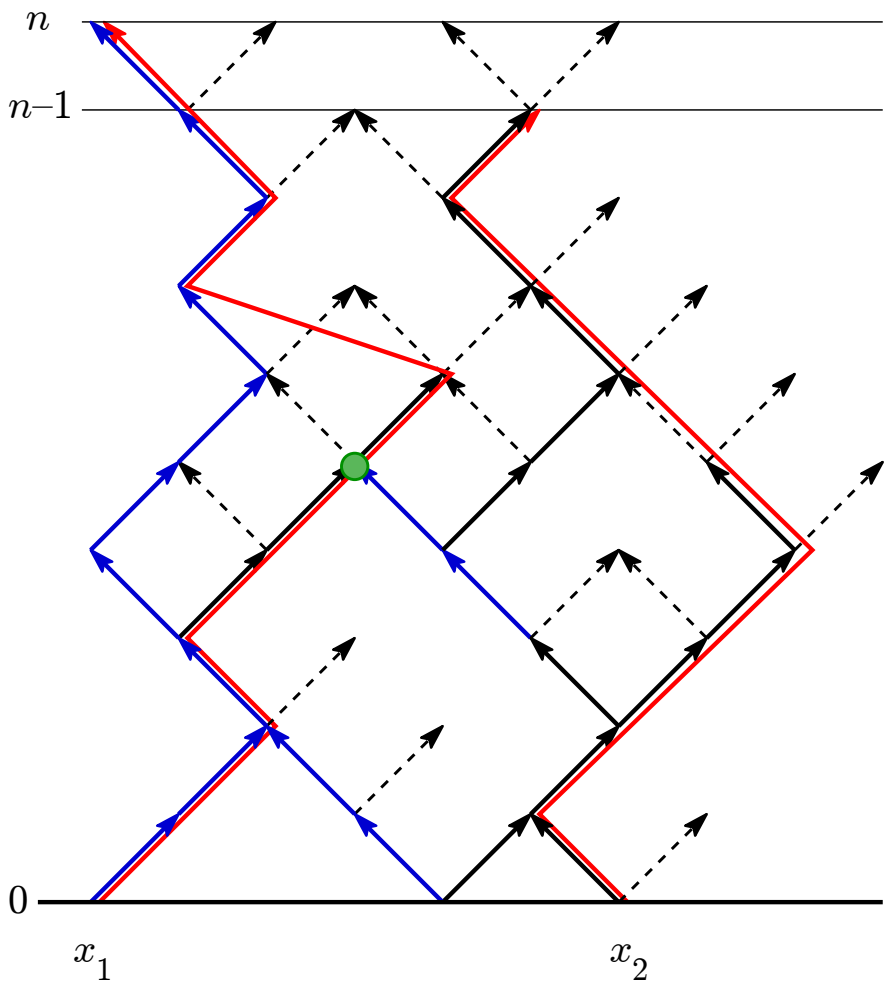
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

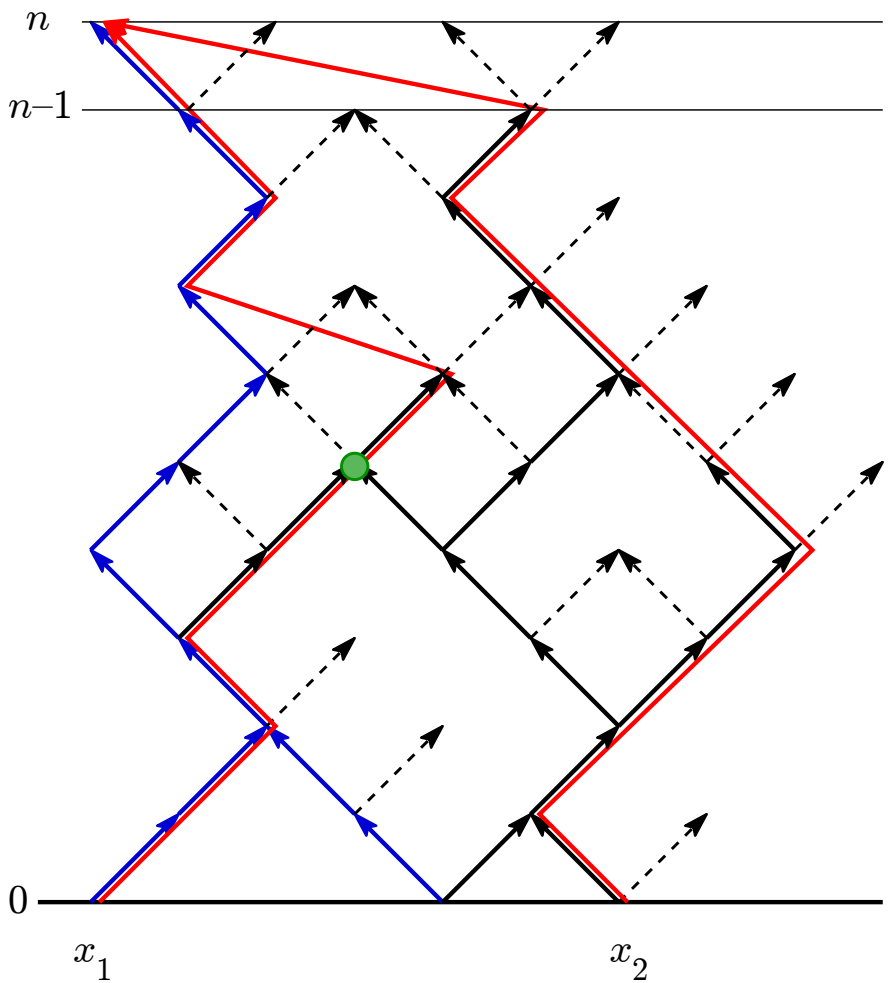
If two exploration clusters start at the same time:



When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

If two exploration clusters start at the same time:

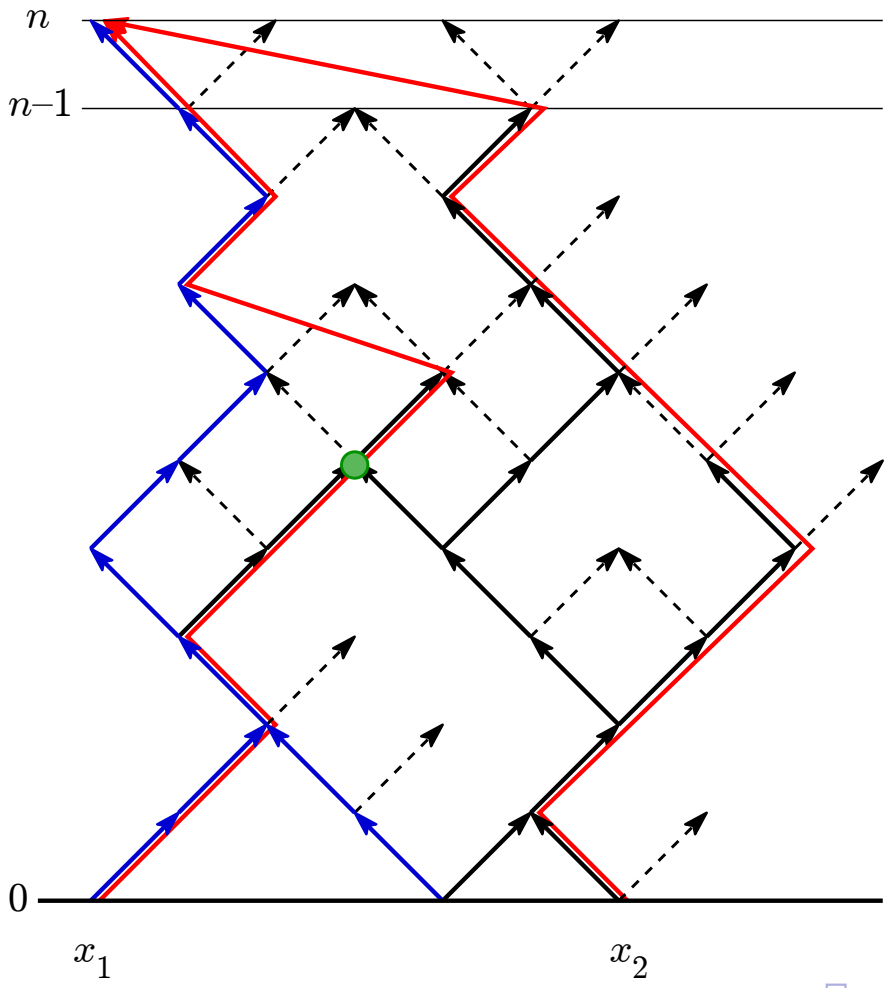


When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

If two exploration clusters start at the same time:

- The boundaries of the two exploration clusters coincide on $[n, m]$ for all times $m \geq n$.

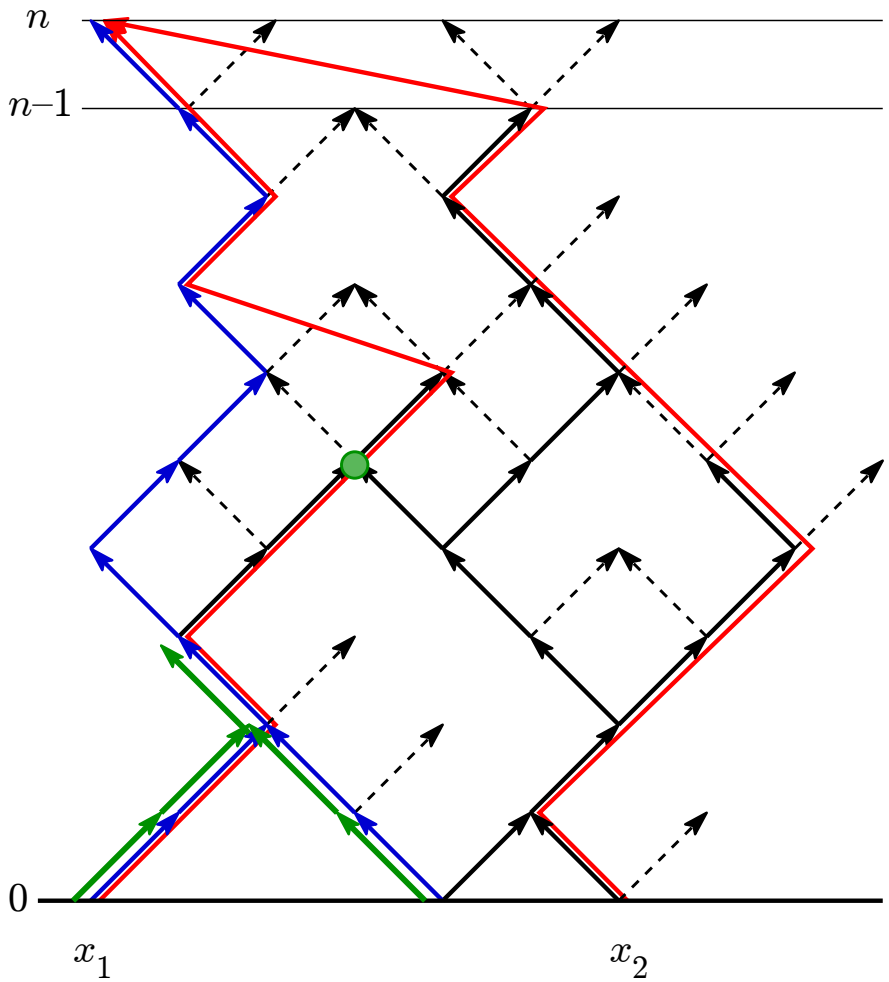


When Two Exploration Clusters Intersect

Different exploration clusters evolve independently before intersecting.

If two exploration clusters start at the same time:

- The boundaries of the two exploration clusters coincide on $[n, m]$ for all times $m \geq n$.
- $\gamma_{(x_1,0)}$ and $\gamma_{(x_2,0)}$ must coalesce before or at time n.



Convergence Criteria (for non-crossing paths)

Convergence to $\mathcal{W}$ follows by verifying these conditions [FINR04]:

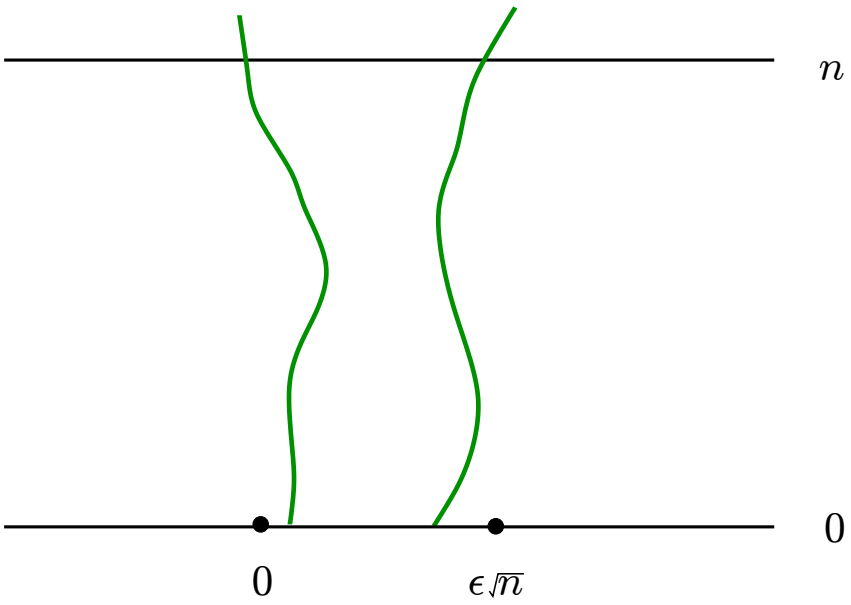
- (FDD) Convergence of a finite number of paths in $S_n\Gamma$ to coalescing Brownian motions.
- (Density Bounds)

Convergence Criteria (for non-crossing paths)

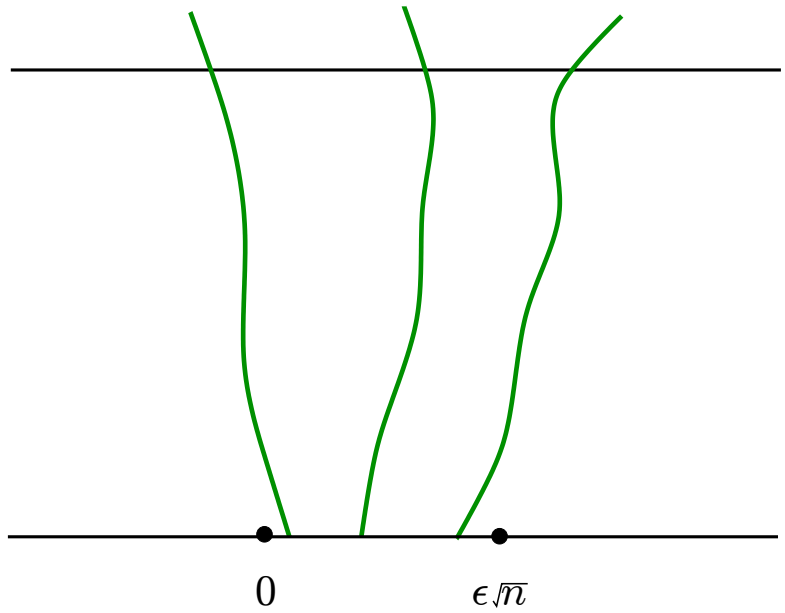
Convergence to $\mathcal{W}$ follows by verifying these conditions [FINR04]:

- (FDD) Convergence of a finite number of paths in $S_n\Gamma$ to coalescing Brownian motions.
- (Density Bounds)

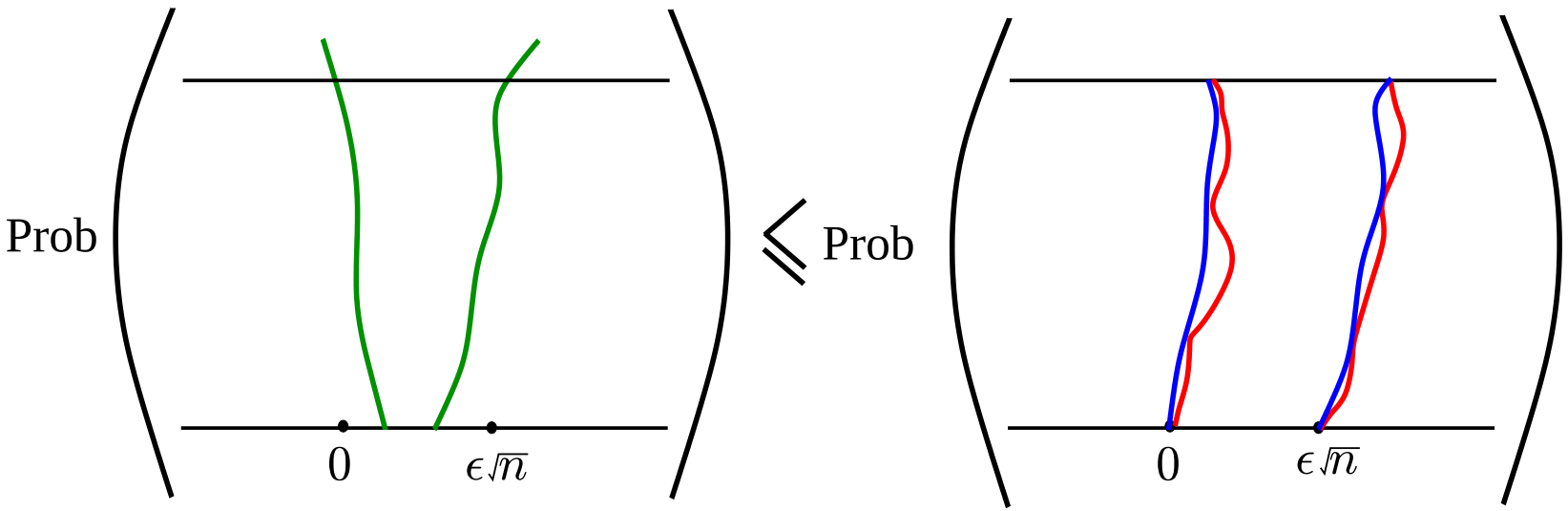
(B1) $\limsup_{n \to \infty} \text{Prob}(\cdot) = O(\epsilon)$



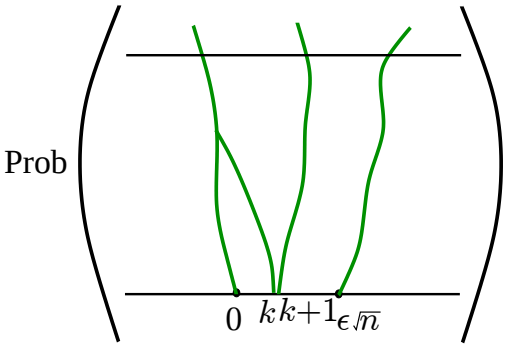
(B2) $\limsup_{n \to \infty} \text{Prob}(\cdot) = O(\epsilon^2)$



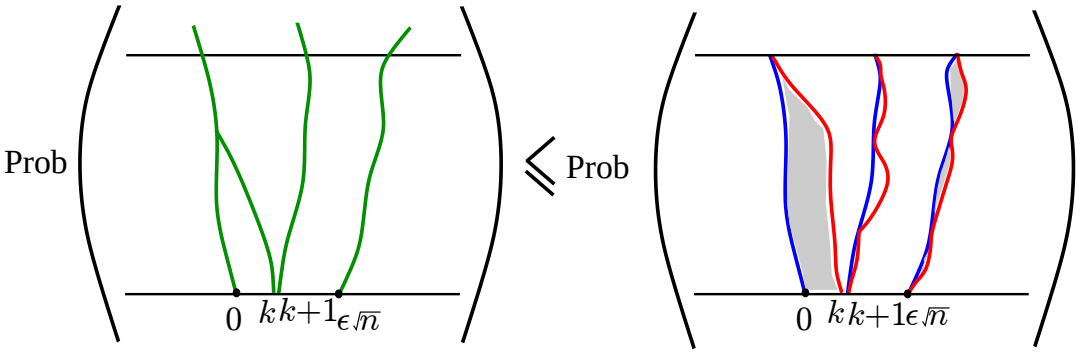
Density Bound (B1)



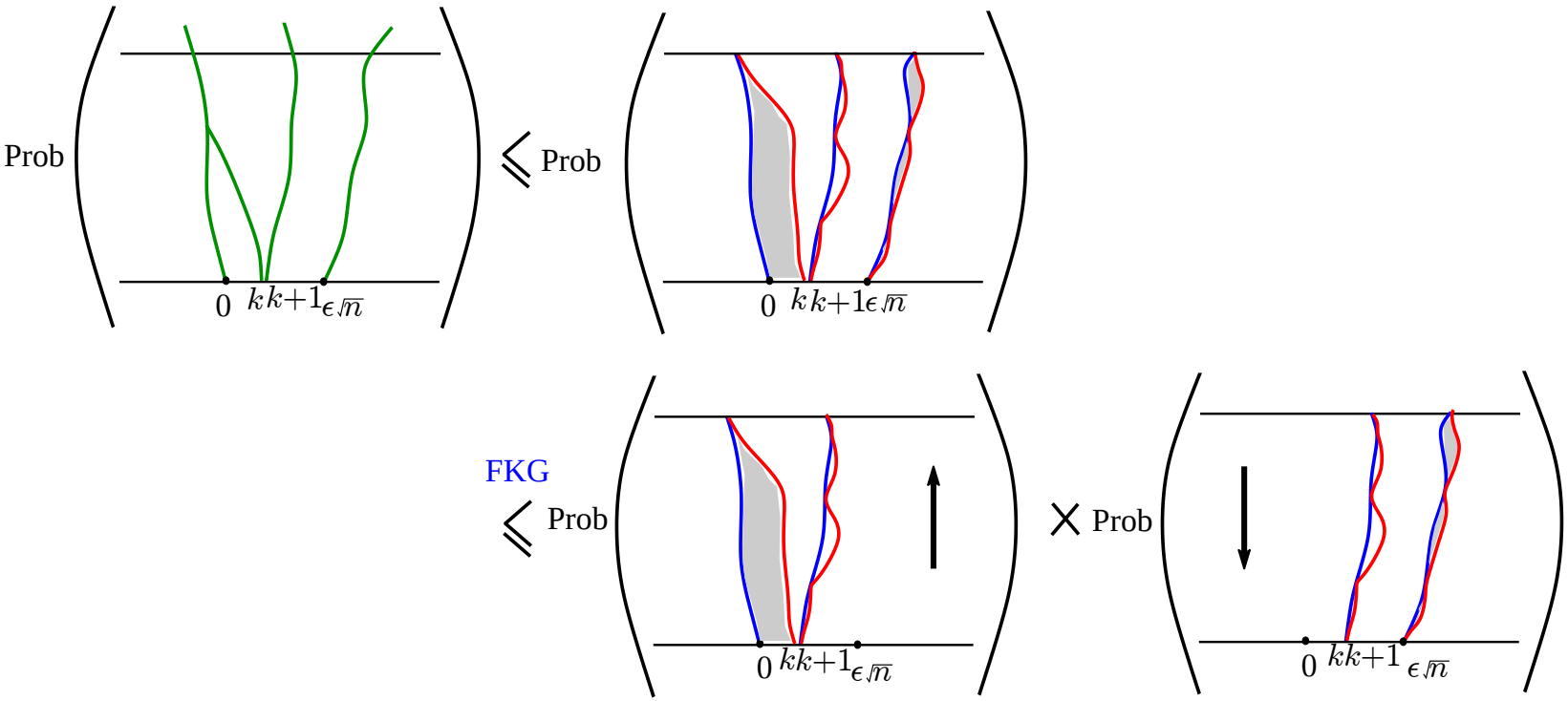
Density Bound (B2)



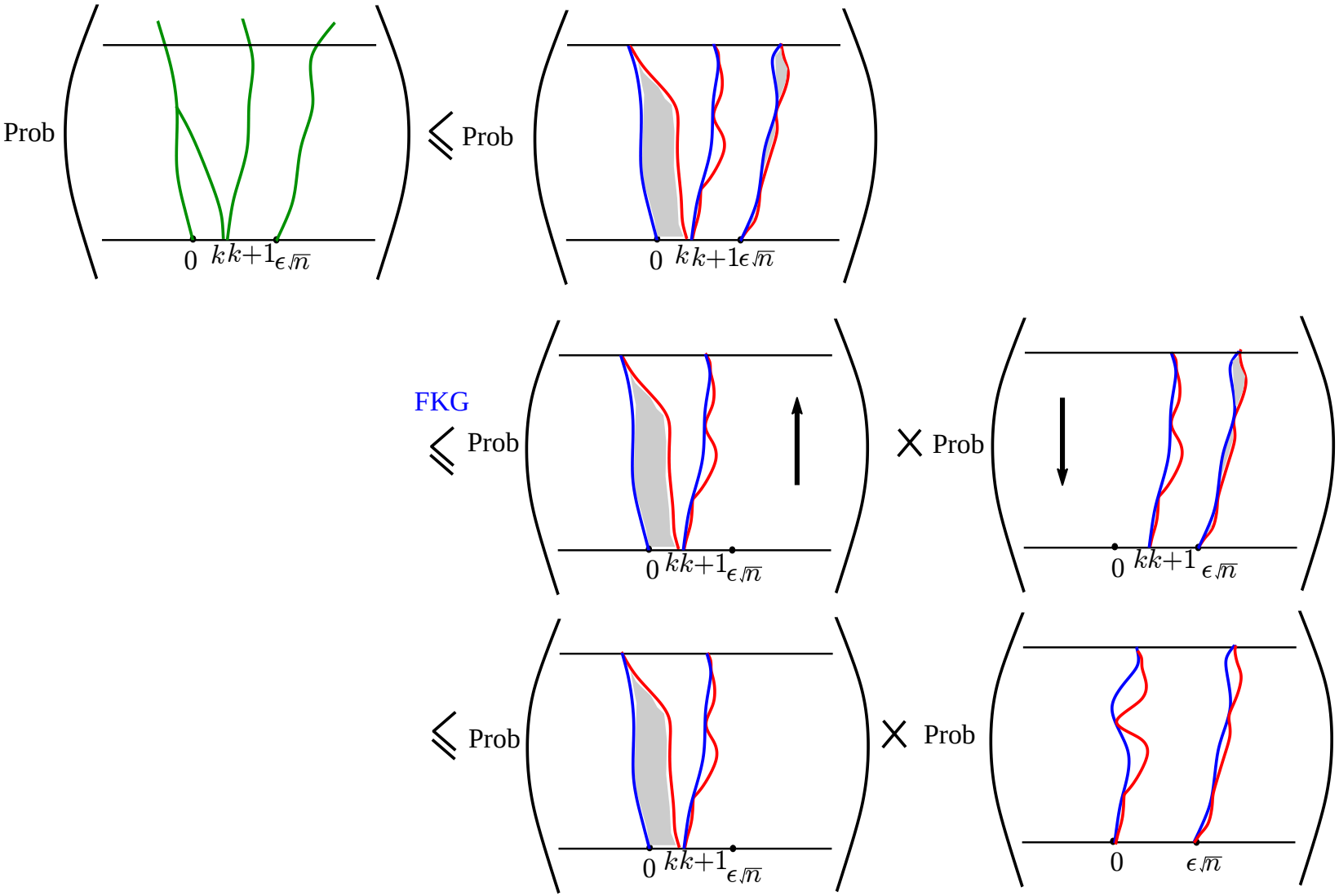
Density Bound (B2)



Density Bound (B2)



Density Bound (B2)



Thank You