On random symmetrizations of convex bodies

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Rouen
2012

Let A be a convex body in $\mathbb{R}^d$, $u \in \mathbb{S}^{d-1}$. Steiner symmetrization

$$A \to S_u A$$

is defined by ...

Properties :

1) $S_u A$ is convex.

2) $\mathrm{Vol}(S_u A) = \mathrm{Vol}(A)$.

3) $\mathrm{Mes}(\partial(S_u A)) \leq \mathrm{Mes}(\partial(A))$.

Let A be a convex body in $\mathbb{R}^d$, $u \in \mathbb{S}^{d-1}$.

Minkowski (or Blachke) symmetrization $B_u A$ is defined by

$$B_u A = \frac{1}{2}(A \bigoplus \pi_u A),$$

where $\bigoplus$ is Minkowski addition, π_u is the symmetry with respect to $u^\perp$.

Properties :

1) $B_u A$ is a convex body.

2) Let f_A be the support function of A : $f_A(\theta) = \sup_{x \in A} \langle \theta, x \rangle$, $\theta \in \mathbb{S}^{d-1}$.
Let $M^\star(A) = \int_{\mathbb{S}^{d-1}} f_A d\sigma$, where σ is the normalized Haar measure on $\mathbb{S}^{d-1}$.
Then
$$M^\star(B_u A) = M^\star(A).$$

3) $\mathrm{Vol}(B_u A) \geq \mathrm{Vol}(A)$.

4) $S_u A \subset B_u A$.

Let $\{u_k\}$, $u_k \in \mathbb{S}^{d-1}$ be a sequence of directions.

Let

$$S_n A = S_{u_n}(S_{u_{n-1}} \dots (S_{u_1} A) \dots),$$

$$B_n A = B_{u_n}(B_{u_{n-1}} \dots (B_{u_1} A) \dots).$$

Asymptotic behaviour of $S_n A$ and $B_n A$ when $n \to \infty$?

In particular, what can we observe if $\{u_k\}$ is a sequence of i.i.d. directions uniformly distributed on $\mathbb{S}^{d-1}$?

Intuitively it seems credible to expect that these sequences will round up and will converge to limit balls.

4. Known results

Theorem A (Mani-Levitska, 1986)

Let A be a convex body in $\mathbb{R}^d$, $\ Vol(A) = Vol(D)$, $\ D = B(0,1)$.
Let $\{u_k\}$ be i.i.d. uniformly distributed directions.
Then a.s.

$$d_H(S_n A,\ D) \to 0,$$

where d_H is Hausdorff distance.

Corollary. Solution of isoperimeter problem.

Theorem B (Klartag, 2003)

Let A be a convex body in $\mathbb{R}^d$. Denote $L = M^\star(A)$.
Without loss of generality we can suppose that $Vol(A) = Vol(D)$.

Then

1) $\exists\, C, C' > 0$ *and a sequence* $\{u_k\}$ *of directions such that*

$$d_H(B_n A,\ LD) \le C \exp\{-C' n\}.$$

2) $\exists\, C, C' > 0$ *and a sequence* $\{u_k\}$ *of directions such that*

$$d_H(S_n A,\ D) \le C \exp\{-C' \sqrt{n}\}.$$

Theorem 1.

Let $\{u_k\}$ *be a sequence of directions such that*
1) for all convex body A

$$d_H(S_nA,\, D_A) \to 0,$$

where D_A *is the centered ball such that* $Vol(D_A) = Vol(A)$.

Then

$$d_H(B_nA,\, M^*D) \to 0.$$

2) If for each convex body A

$$d_H(B_nA,\, M^*A) \to 0,$$

Then

$$d_H(S_nA,\, D_A) \to 0.$$

7. Our results

Theorem 2.

Let A be a convex body in $\mathbb{R}^d$, $L = M^\star(A)$. Let $\{u_k\}$ be i.i.d. directions with common distribution μ.

Suppose that $\mu \ll \sigma$ *and for σ-almost all* $\theta \in \mathbb{S}^{d-1}$

$$\frac{d\mu}{d\sigma}(\theta) \le a < \frac{d}{d-1}.$$

Then $\exists\, C, C' > 0$ *such that with probability 1 for $n \ge n(\omega)$*

$$d_H(B_n A,\ LD) \le C \exp\{-C' n\}.$$

Comments

a) $d = 2$
b) $d \ge 3$

8. Our results

Theorem 3.

Let A be a convex body in $\mathbb{R}^d$, $Vol(A) = Vol(D)$, $D = B(0,1)$.
Let $\{u_k\}$ be i.i.d. directions with common distribution μ.
Suppose that $\mu \ll \sigma$ and for σ-almost all $\theta \in \mathbb{S}^{d-1}$

$$\frac{d\mu}{d\sigma}(\theta) \leq a < \frac{d}{d-1}.$$

Then $\exists\, C, C' > 0$ such that with probability 1 for $n \geq n(\omega)$

$$d_H(S_n A,\ D) \leq C \exp\{-C' \sqrt{n}\}.$$

Comments

Passage from $B_n A$ to $S_n A$.

Open questions

1. Exponential rate for $S_n A$.
2. Conditions on μ.
3. Nonrandom asymptotically uniformly distributed directions.

10. References

1. P. Mani-Levitska, Random Steiner symmetrizations, Stud.Scient.Math.Hungarica, 1986, v. 21, pp. 373-378.
2. B. Klartag, Rate of convergence of geometric symmetrizations, GAFA, 2004, v. 21, N 6, pp. 1322-1338.