# Continuum percolation and phase transition for multi-type Quermass Model

**David Dereudre**,
Laboratoire Paul Painlevé, University Lille 1
(Joint work with **David Coupier**)

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# State spaces

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- $\pi^z$ the Poisson Point Process on $\mathcal{E}$ with intensity $z\lambda \otimes Q \otimes \mathcal{U}_K$ where $z > 0$, $Q$ probability measure on $\mathbb{R}^+$ and $\mathcal{U}_k$ the uniform distribution on $\{1, \ldots, K\}$.

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- $\pi^z_\Lambda$ the Poisson Point Process on $\mathcal{E}_\Lambda := \Lambda \times \mathbb{R}^+ \times \{1, \ldots, K\}$ with intensity $z\lambda_\Lambda \otimes Q \otimes \mathcal{U}_K$.

# Multi-type Boolean model

- **Non-overlapping density :**
  For any configuration $\gamma$ in $\mathcal{E}_\Lambda$

$$f_\Lambda(\gamma) = 1\!\!1_{\mathcal{A}_{\text{nob}}}(\gamma),$$

where $\gamma \in \mathcal{A}_{\text{nob}} \subset \mathcal{M}(\mathcal{E})$ if and only if
$\forall (x, R, k), (x', R', k') \in \gamma$

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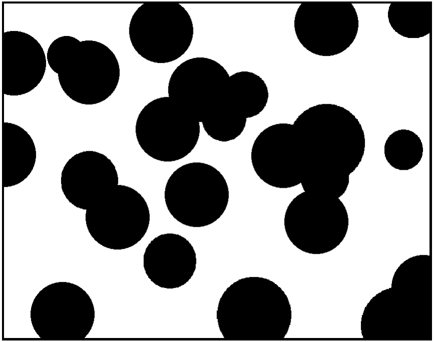
  -On $\mathbb{R}^2$ : Gibbs (or Markov) modifications. We can prove that there exists a such model.

# Representation of the multi-type Boolean model on $\Lambda$

For $K=2$ : red particle $(k=1)$ and blue particle $(k=2)$.
A one-type Boolean model on $\Lambda$ with density $2^{N_{cc}(\bar{\gamma})}$ :

$$Q_\Lambda(d\gamma) = \frac{1}{Z_\Lambda} 2^{N_{cc}(\bar{\gamma})} \pi_\Lambda^z(d\gamma).$$



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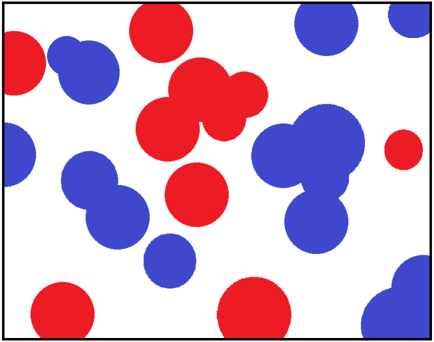

In colouring independently the connected components, we obtain a 2-type boolean model on $\Lambda$.

# The multi-type Quermass model on $\Lambda$

- **Quermass density :**
  For any configuration $\gamma$ in $\mathcal{E}_\Lambda$

$$H(\gamma) = \theta_1 \mathcal{A}(\bar{\gamma}) + \theta_2 \mathcal{L}(\bar{\gamma}) + \theta_3 \chi(\bar{\gamma}),$$

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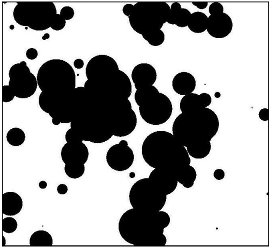
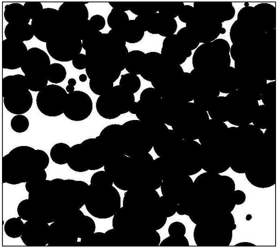
- **References :**
  - $K = 1$, $\theta_1 \geq 0$, $\theta_2 = \theta_3 = 0$ : Widom - Rowlinson (70)
  - $K = 1$ : Likos - Mecke - Wagner (95), Mecke (96), Kendall - Van Lieshout - Baddeley (99).
  - $K > 1$, $\theta_1 \geq 0$, $\theta_2 = \theta_3 = 0$ : Chayes - Kotechy (95), Giacomin - Lebowitz - Maes (95)

# Exemples of one-type Quermas models

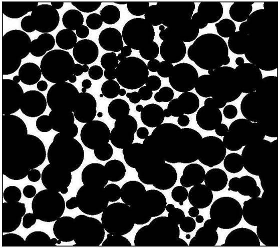
For $K = 1$ :



$\theta_1 = 0,\ \theta_2 = 0.2,\ \theta_3 = 0$



$\theta_1 = 0,\ \theta_2 = 0,\ \theta_3 = 1$



$\theta_1 = -1,\ \theta_2 = -1,\ \theta_3 = 0$

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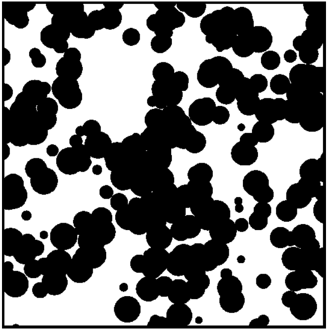
Example with $\theta_1 = -0.2$, $\theta_2 = 0.3$ and $\theta_3 = 0$ :



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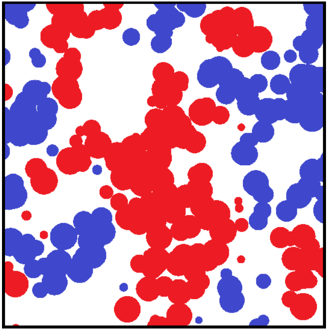

In colouring independently the connected components, we obtain a 2-type Quermass model on $\Lambda$ for the same parameters.

# Multi-type Quermass models in $\mathbb{R}^2$

We assume that $Q([0, R_0]) = 1$.

- **The local Quermass conditional density :**

$$H_\Lambda(\gamma_\Lambda | \gamma_{\Lambda^c}) = H(\gamma_{\Lambda \oplus B(0,2R_0)}) - H(\gamma_{\Lambda \oplus B(0,2R_0) \setminus \Lambda}).$$

$$f_\Lambda(\gamma_\Lambda | \gamma_{\Lambda^c}) = \mathbb{1}_{\mathcal{A}_{\text{nob}}}(\gamma) e^{-H_\Lambda(\gamma_\Lambda | \gamma_{\Lambda^c})}.$$

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- **Questions :** Existence, uniqueness, non uniqueness (phase transition), percolation ?

# First results.

- **Widom-Rowlinson** (70) : $K = 1$, $\theta_1 \geq 0$, $\theta_2 = 0$, $\theta_3 = 0$. Existence and phase Transition for $z$ large enough.
- **Chayes-Kotecky** (95) : $K \geq 1$, $\theta_1 \geq 0$, $\theta_2 = 0$, $\theta_3 = 0$ Percolation and phase transition for $z$ large enough.
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Percolation for one-type Quermass Model with $z$ large enough

$$\Downarrow$$

Phase transition for multi-type Quermass model $(K > 1)$

with $z$ large enough.

2. Percolation.

# Main Result

In this Section $K = 1$
"$\bar{\gamma}$ percolates" means "there exists an unbounded connected component in $\bar{\gamma}$".

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**Theorem (Coupier, Der.)**

*We assume that $Q([R_0, R_1]) = 1$ with ($R_0 > 0$ and $R_1 < \infty$), then for any coefficients $\theta_1, \theta_2, \theta_3$ in $\mathbb{R}$, there exists $z^* > 0$ such that for any $z > z^*$ and any Quermass process $P$ for parameters $z$, $\theta_1, \theta_2, \theta_3$,*

$$P(\bar{\gamma} \text{ percolates}) = 1,$$

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**Main issue :** when $\theta_3 \neq 0$, it is impossible to obtain a stochastic minoration of $P$ by Poisson processes

$$\text{For all } z' > 0, \qquad \pi_\Lambda^{z'} \not\preceq P_\Lambda.$$

# The connection Lemma

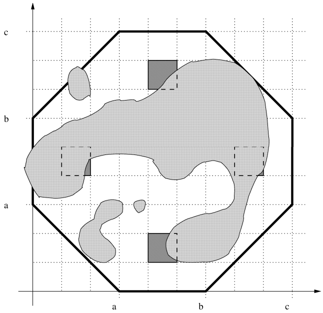
$\mathcal{D}$ = the diamond box

$\mathcal{D}$ is open for $\bar{\gamma}$ if

a) $\bar{\gamma} \cap B_N \neq \emptyset$

b) the same for $B_E, B_W, B_S$

c) $B_N, B_E, B_W, B_S$ are connected via $\bar{\gamma}_{\mathcal{D}}$




## Lemma (Connection Lemma)

*There exists $C > 0$ (depending on $\theta_1$, $\theta_2$ and $\theta_3$) such that for any $z > 0$ and any Quermass process $P$*

$$\inf_{\gamma_{\Lambda^c}} P(\mathcal{D} \textit{ is open } |\gamma_{\Lambda^c}) \geq 1 - \frac{C}{z}.$$

# Classical Bernoulli domination

Let $(V, E)$ be an undirected graph with uniformly bounded degrees and $\xi$ a random variable in $\{0,1\}^V$

**Lemme (Liggett et al. 97)**

*Let $p \in [0,1]$. Assume that for all $x \in V$,*

$$P\left(\xi_x = 1 \,|\, \xi_y : \{x,y\} \notin E\right) \geq p \quad a.s.$$

*Then the law of $\{\xi_x, x \in V\}$ dominates stochastically a product $\otimes_{x\in\mathcal{V}} B_x$ of Bernoulli laws with parameter $f(p)$, with $\lim_{p\to 1} f(p) = 1$.*

## 3 Phase transition

# Results

**Theorem (Coupier, Der.)**

*We assume that* $Q([R_0, R_1]) = 1$ *with* $(R_0 > 0$ *and* $R_1 < \infty)$, *then for any coefficients* $\theta_1, \theta_2, \theta_3$ *in* $\mathbb{R}$ *and* $K > 1$, *there exists* $z^* > 0$ *such that for any* $z > z^*$, *there exist several multi-type Quermass processes for parameters* $z$, $\theta_1, \theta_2, \theta_3$ *and* $K$.

In particular, there exist several multi-type Boolean models for $K$ fixed and $z$ large enough.

# Approximation by finite volume multi-type Quermass processes

## Proposition

*The set of multi-type Quermass processes is a Choquet's Simplex such that each extremal process $P$ is ergodic with the following approximation*

$$P = \lim_{\Lambda \to \mathbb{R}^2} P(.|\gamma_{\Lambda^c}),$$

*for $P$ almost every $\gamma$.*
*Conversely, for every $\gamma$ such that the multi-type Quermass process on $\Lambda$ given the outside configuration $\gamma_{\Lambda^c}$ converges to a probability measure $P$. Then $P$ is a multi-type Quermass process on $\mathbb{R}^2$.*

# Stochastic domination for $2^{N_{cc}} P_\Lambda$

**Proposition**

*Let $\gamma$ be a configuration. Let $(P_\Lambda^z(.|\gamma_{\Lambda^c}))_{z>0}$ be the one-type Quermass processes on $\Lambda$ for parameters $z$ and $\theta_1$, $\theta_2$, $\theta_3$. Then*

$$P_\Lambda^{Cz}(.|\gamma_{\Lambda^c}) \preceq \frac{1}{Z_\Lambda} 2^{N_{cc}} P_\Lambda^z(.|\gamma_{\Lambda^c}),$$

*with*

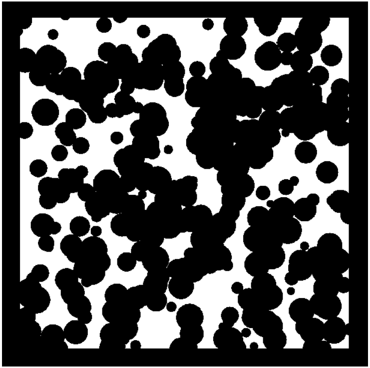
$$C = 2^{-\frac{\pi}{arcsin(\frac{R_0}{R_0+R_1})}} > 0$$

.

In particular, thanks to the approximation and the representation of the multi-type Quermass processes, we show that the multi-type Quermass process percolates for $z$ large enough.

# The phase transition proof

Let $\frac{1}{Z_\Lambda} 2^{N_{cc}} P^z_\Lambda(.|\gamma_{\Lambda^c})$ be a modified one-type Quermass process with a full boundary condition



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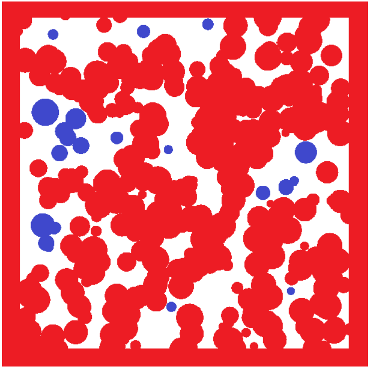

2-type Quermass Process in $\Lambda$ with red boundary condition

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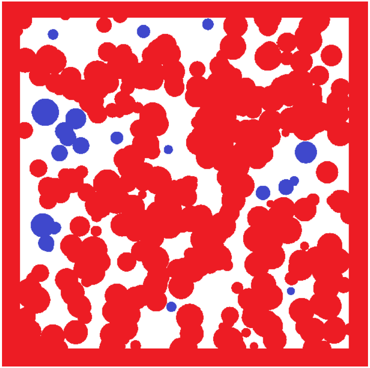

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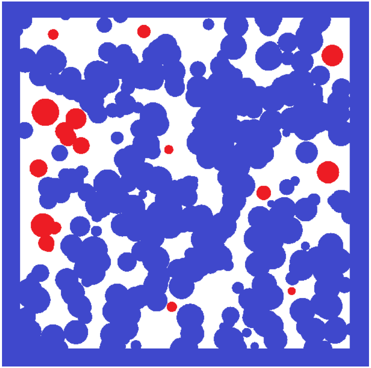

2-type Quermass Process in $\Lambda$ with blue boundary condition

# The phase transition proof

- When $\Lambda$ goes to $\mathbb{R}^2$, the 2-type Quermass process in $\Lambda$ with red boundary condition goes to a 2-type Quermass process in $\mathbb{R}^2$ with the red particle density bigger than the blue particle density (if percolation occurs).

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- Conversely for the 2-type Quermass process in $\Lambda$ with blue boundary condition.
- We build two different 2-type Quermass processes in $\mathbb{R}^2$.