On comparison of clustering properties of point processes

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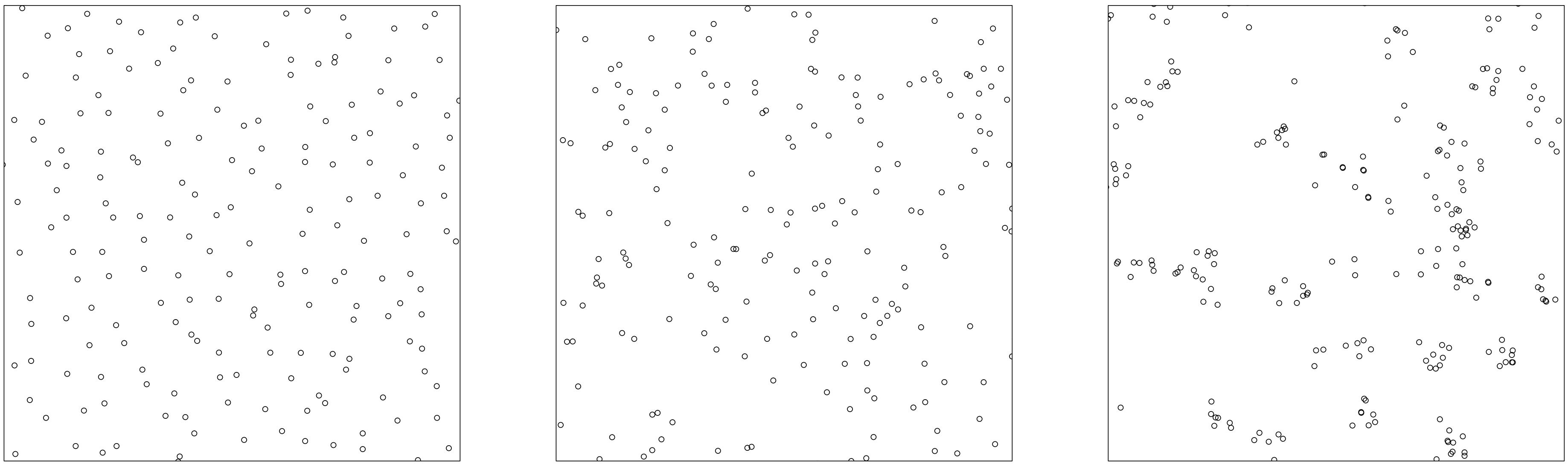

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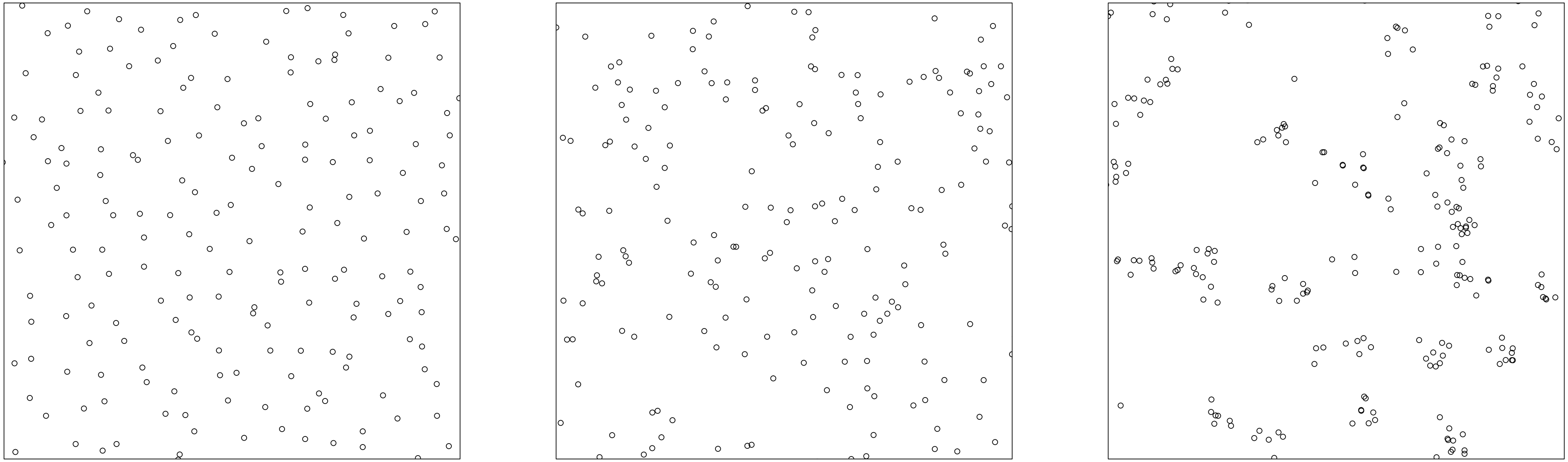

How to compare clustering properties of two point processes (pp) Φ_1, Φ_2 having “on average” the same number of points per unit of space?

More precisely, having the same mean measure:
$\mathbf{E}(\Phi_1(B)) = \mathbf{E}(\Phi_2(B))$ for all $B \subset \mathbb{E}$.

Motivation

Develop models and tools for the study of impact of clustering of nodes on the performance of geometric networks.

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Develop models and tools for the study of impact of clustering of nodes on the performance of geometric networks.

Particular application domain — wireless networks.

We are interested in

- coverage, (SINR coverage, throughput),
- percolation (connectivity),
- first passage percolation (routing),
- ...

Some references

- I. Benjamini and A. Stauffer (2011) Perturbing the hexagonal circle packing: a percolation perspective. arXiv
- M. Franceschetti, L. Booth, M. Cook, R.W. Meester, and J. Bruck (2005) Continuum percolation with unreliable and spread-out connections. J. Stat. Phy.
- M. Franceschetti, M. Penrose, and T. Rosoman. (2010) Strict inequalities of critical probabilities on Gilbert's continuum percolation graph. arXiv
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- R. Roy and H. Tanemura (2002) Critical intensities of boolean models with different underlying convex shapes. Ann. Appl. Probab.

Clustering comparison tools

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$\Rightarrow$ Comparisons of void probabilities and all higher-order factorial moment measures. Statistically larger voids and moments — more clustering.

$\Rightarrow$ dcx ordering of pp. Natural extension of dcx ordering of random vectors (recall Ross's conjecture), a generalization of convex ordering of random variables. Larger in dcx pp represent more variability (in probability and in state space — clustering).

dcx (directionally convex) functions

Function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ twice differentiable is dcx if $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0$ for all $x \in \mathbb{R}^d$ and $\forall i, j$.

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Definition can be extended to all functions by saying that f is dcx if all difference operators $\Delta_i^\delta f(x) := f(x + \delta e_i) - f(x)$ are non-negative; $\Delta_i^\epsilon \Delta_j^\delta f(x) \geq 0$, $\forall x \in \mathbb{R}^d, i, j \in \{1, \ldots, d\}$, $\delta > 0, \epsilon > 0$.

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Examples:

- $f(x) = e^{-\sum_i a_i x_i}$, $a_i \geq 0$.
- $f(x) = \prod_i \max(x_i, a_i)$, a_i constants,

dcx ordering of point processes

Define: $\Phi_1 \leq_{dcx} \Phi_2$ if for all bounded Borel subsets $B_1, \ldots, B_n$,

$$\Big(\Phi_1(B_1), \ldots, \Phi_1(B_n)\Big) \leq_{dcx} \Big(\Phi_2(B_1), \ldots, \Phi_2(B_n)\Big);$$

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i.e, $\forall f$ dcx, bounded Borel subsets $B_1, \ldots, B_n$,

$$\mathsf{E}\Big(f\Big(\Phi_1(B_1), \ldots, \Phi_1(B_n)\Big)\Big) \leq \mathsf{E}\Big(f\Big(\Phi_2(B_1), \ldots, \Phi_2(B_n)\Big)\Big).$$

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Sufficient condition: Enough to verify the inequality on disjoint bounded Borel subsets (bBs).

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dcx is a partial order (reflective, antisymmetric and transitive) of point process with locally finite mean measure (to ensure transitivity).

dcx for point processes; properties

- If $\Phi_1 \le_{dcx} \Phi_2$ then Φ_1 and Φ_2 have equal mean measures; $\mathsf{E}(\Phi_1(\cdot)) = \mathsf{E}(\Phi_2(\cdot))$.

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- If $\Phi_1 \le_{dcx} \Phi_2$ then Φ_1 and Φ_2 have equal mean measures; $\mathbf{E}(\Phi_1(\cdot)) = \mathbf{E}(\Phi_2(\cdot))$.
- dcx is preserved by independent thinning, marking and superpositioning; i.e.,

$$\text{If } \Phi_1 \le_{dcx} \Phi_2 \text{ then } \tilde{\Phi}_1 \le_{dcx} \tilde{\Phi}_2 \,,$$

where $\tilde{\Phi}_i$ is a version of Φ_i independently thinned (or marked, or superposed with a given point process).

dcx and shot-noise fields

Given point process Φ and a non-negative function $h(x,y)$ on $(\mathbb{R}^d, S)$, measurable in x, where S is some set, define shot noise field: for $y \in S$

$$V_\Phi(y) := \sum_{X \in \Phi} h(X,y) = \int_{\mathbb{R}^d} h(x,y)\Phi(dx)\,.$$

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Proposition. [BB-Yogesh'09] *If $\Phi_1 \le_{dcx} \Phi_2$ then*

$$\Big(V_{\Phi_1}(y_1),\ldots,V_{\Phi_1}(y_n)\Big) \le_{dcx} \Big(V_{\Phi_2}(y_1),\ldots,V_{\Phi_2}(y_n)\Big)$$

for any finite subset $\{y_1,\ldots,y_n\} \subset S$, provided the RHS has finite mean. In other words, dcx is preserved by the shot-noise field construction.

dcx and shot-noise fields; cont'd

Proof.

- Approximate the integral by simple functions as usual in integration theory: $a.s.$ and in L_1
  $\sum_{i=1}^{k_n} a_{in}\Phi(B_{in}^j) \to \int_{\mathbb{R}^d} h(x,y)\Phi(dx) = V_\Phi(y_j)$, $a_{in} \geq 0$.

dcx and shot-noise fields; cont'd

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- Increasing linear operations preserve dcx hence approximating simple functions are dcx ordered.

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- Increasing linear operations preserve dcx hence approximating simple functions are dcx ordered.
- dcx order is preserved by *joint* weak and L_1 convergence. Hence limiting shot-noise fields are dcx ordered.

dcx and extremal shot-noise fields

In the setting as before define for $y \in S$

$$U_\Phi(y) := \sup_{X \in \Phi} h(X, y)\,.$$

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Proposition. [BB-Yogesh'09] *If* $\Phi_1 \le_{dcx} \Phi_2$ *then for all* $y_1, \dots, y_n \in S$*;* $a_1, \dots, a_n \in \mathbb{R}$*,*
$\mathsf{P}\Big(U_{\Phi_1}(y_i) \le a_i, 1 \le i \le m\Big) \le \mathsf{P}\Big(U_{\Phi_2}(y_i) \le a_i, 1 \le i \le m\Big)$;
i.e, the (joint) finite-dimensional distribution functions of the extremal shot-noise fields are ordered (lower orthant order).

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i.e, the (joint) finite-dimensional distribution functions of the extremal shot-noise fields are ordered (lower orthant order).

Corollary. *One-dimensional distributions of the extremal shot-noise fields are strongly ordered with reversed inequality $U_{\Phi_2}(y) \le_{st} U_{\Phi_1}(y)$, $\forall y \in S$.*

dcx and extremal shot-noise fields; cont'd

Proof.

- Reduction to an (additive) shot noise:

$$\begin{aligned}
&\mathbf{P}\left(U_\Phi(y_i) \le a_i, 1 \le i \le n\right) \\
&\quad = \mathbf{E}\left(e^{-\sum_{i=1}^{n}\sum_{X\in\Phi} -\log \mathbf{1}[h(X,y_i)\le a_i]}\right).
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- $e^{-\sum x_i}$ is dcx function.

Outline of the remaining part of the talk

- $\Rightarrow$ Clustering — models,
- Clustering and coverage,
- $\Rightarrow$ Clustering and percolation,
- Clustering and first passage percolation.

Clustering — models

Poisson point process

Given deterministic, locally finite measure $\Lambda(\cdot)$ on $\mathbb{E} = \mathbb{R}^d$.

Definition. $\Phi = \Phi_\Lambda$ is Poisson point process on $\mathbb{E}$ of intensity $\Lambda(\cdot)$ ($Poi(\Lambda)$) if for any $B_1, \dots, B_n$, bounded, pairwise disjoint subset of $\mathbb{E}$

- $\Phi(B_1), \dots, \Phi(B_n)$ are independent random variables and
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Homogeneous case: $\Lambda(\mathrm{d}x) = \lambda \mathrm{d}x$ for some $0 < \lambda < \infty$.

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$\boldsymbol{\Lambda}$ is the mean measure of $\boldsymbol{\Phi}_{\boldsymbol{\Lambda}}$.

Poisson point process, cont'd

- Void probabilities:

$$\nu_\Phi(B) = \mathsf{P}\left(\Phi(B) = 0\right) = e^{-\Lambda(B)}.$$

Poisson point process, cont'd

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$$\nu_\Phi(B) = \mathsf{P}\left(\Phi(B) = 0\right) = e^{-\Lambda(B)}\,.$$

- Moment measure of order k:

$$\alpha^{(k)}(B_1 \times \ldots \times B_k) = \mathsf{E}\left(\prod_{i=1}^{k} \Phi(B_i)\right) = \prod_{i=1}^{k} \Lambda(B_i)$$

for mutually disjoint $B_1, \ldots, B_k$

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- In Homogeneous case: Riplay's function $K(r) \equiv \pi r^2$ and pair correlation function $g(x) \equiv 1$.

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Given random, locally finite measure $\boldsymbol{\mathcal{L}}(\cdot)$ on $\mathbb{E} = \mathbb{R}^{\boldsymbol{d}}$.

Definition. $\boldsymbol{\Phi_{\mathcal{L}}}$ is Cox point process on $\mathbb{E}$ of intensity $\boldsymbol{\mathcal{L}}(\cdot)$ ($\boldsymbol{Cox}(\boldsymbol{\mathcal{L}})$) if conditionally, given $\boldsymbol{\mathcal{L}}(\cdot) = \boldsymbol{\Lambda}(\cdot)$, $\boldsymbol{\Phi_{\mathcal{L}}}$ is Poisson point process with intensity measure $\boldsymbol{\Lambda}$.

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- $\mathbf{P}\left(\Phi_{\mathcal{L}}(B) = 0\right) = \mathbf{E}(\mathbf{P}\left(\Phi_{\Lambda}(B) = 0 | \mathcal{L} = \Lambda\right)) = \mathbf{E}\left(e^{-\Lambda(B)} | \mathcal{L} = \Lambda\right) \le e^{-\mathbf{E}(\mathcal{L}(B))}$ (Jensen's inequality). Hence, void probabilities of $Cox(\mathcal{L})$ are larger than these of $Poi(\mathbf{E}(\mathcal{L}))$.

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- More assumptions on $\mathcal{L}$ needed to get inequality for moment measures and dcx order.

Super-Poisson pp (cluster more)

strongly ($\boldsymbol{dcx}$-larger) than Poisson

- Poisson-Poisson cluster pp; $\boldsymbol{\mathcal{L}}(\mathrm{d}\boldsymbol{x}) = \sum_{\boldsymbol{Y}\in\boldsymbol{\Psi}} \boldsymbol{\Lambda}(\mathrm{d}\boldsymbol{x} + \boldsymbol{Y})$, where $\boldsymbol{\Psi}$ is a Poisson ("parent") process; (we will show an example)

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- Lévy based Cox pp; $\mathcal{L}(B_1), \ldots, \mathcal{L}(B_n)$ are independent variables for pair-wise disjoint $B_i's$ (complete independence property) [Hellmund, Prokěová, Vedel Jensen'08];

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- some perturbed Poisson pp (to be explained)
- some perturbed lattice pp (to be explained)

Super-Poisson pp (cluster more); cont'd

weakly (voids and moments larger than for Poisson of the same mean)

- (Positively) associated pp:
  Cov $(f(\Phi(B_1),\ldots,\Phi(B_k))g(\Phi(B_1),\ldots,\Phi(B_k))) \geq 0$
  for all $B_1,\ldots,B_k$, $0 f,g \geq 0$ increasing functions;
  [BB-Yogesh'11]
- Cox pp with associated intensity measures; [Waymire'85]
- Permanental processes; density of the k th factorial moment measure is given by
  $\rho^{(k)}(x_1,\ldots,x_k) = \mathrm{per}(K(x_i,x_j))_{1\leq i,j\leq k}$, where per stands for permanent of a matrix and K is some kernel (assumptions needed). It is also a Cox process!;
  [Ben Hough'09, BB-Yogesh'11]

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- processes with some "repulsion mechanism" between points (like some Gibb's point processes)?
- Well..., not immediately. Some (much) extra assumptions and modification are needed.

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strongly (in dcx)

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weakly (voids and moments)

- Negatively associated point processes;
  Cov $(f(\Phi(B_1),\ldots,\Phi(B_k)), g(\Phi(B_{k+1}),\ldots,\Phi(B_l))) \leq 0$
  for all of bBs $B_1,\ldots,B_l$ s.t.
  $(B_1 \cup \ldots \cup B_k) \cap (B_{k+1} \cup \ldots \cup B_l) = \emptyset$ and $0 f, g \geq 0$
  increasing functions; [BB-Yogesh'11]

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  increasing functions; [BB-Yogesh'11]
- Determinantal point processes density of the k th factorial moment measure is given by
  $\rho^{(k)}(x_1,\ldots,x_k) = \det(K(x_i,x_j))_{1\le i,j\le k}$, where det stands for determinant of a matrix and K is some kernel (assumptions needed). It is a Gibbs process!; [Ben Hough'09, BB-Yogesh'11]

More for determinantal and permanental

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It follows for example that, the pp of radii of the Ginibre$(*)$ pp is (dcx) sub-Poisson.

$(*)$ The determinantal pp with kernel $K((x_1, x_2), (y_1, y_2)) = \exp[(x_1 y_1 + x_2 y_2) + i(x_2 y_1 - x_1 y_2)]$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, with respect to the measure $\mu(d(x_1, x_2)) = \pi^{-1} \exp[-x_1^2 - x_2^2]\, dx_1 dx_2$.

Perturbation of a point processes

Φ a pp on $\mathbb{R}^d$, $\mathcal{N}(\cdot,\cdot)$, $\mathcal{X}(\cdot,\cdot)$ be two probability kernels from $\mathbb{R}^d$ to non-negative integers $\mathbb{Z}^+$ and $\mathbb{R}^d$, respectively. Define a new pp on $\mathbb{R}^d$

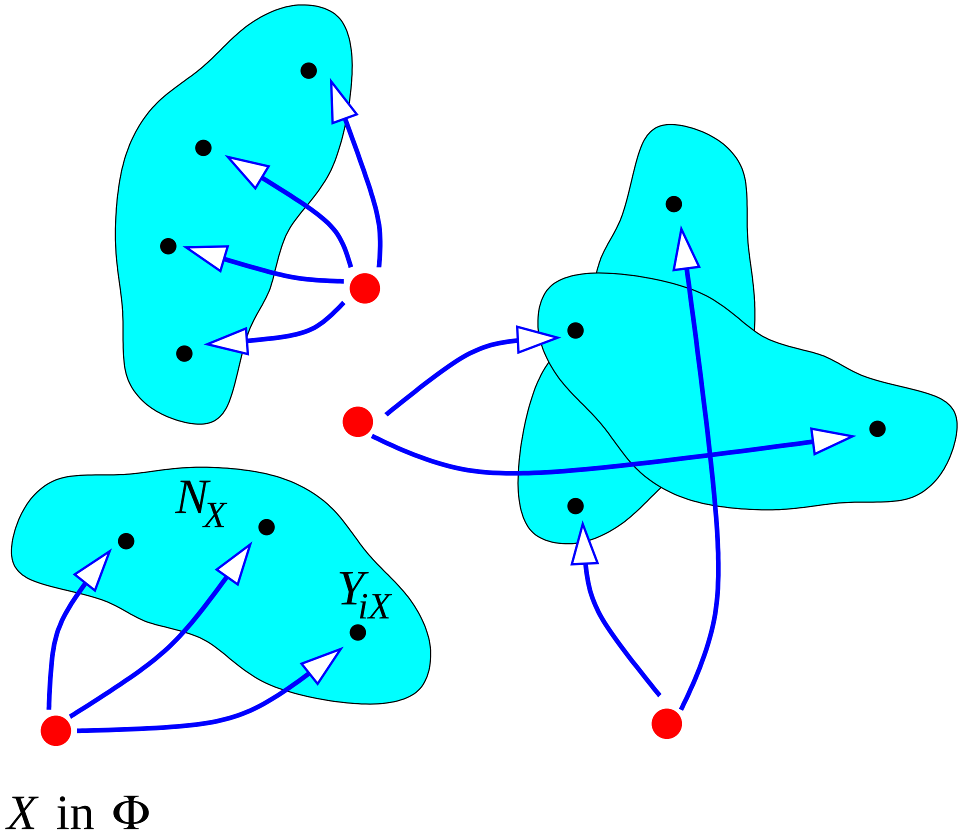
$$\Phi^{pert} := \bigcup_{X\in\Phi} \bigcup_{i=1}^{N_X} \{X + Y_{iX}\},$$

where

- N_X, $X \in \Phi$ are independent, non-negative integer-valued random variables with distribution $\mathbf{P}(N_X \in \cdot \mid \Phi) = \mathcal{N}(X,\cdot)$,
- $\mathrm{Y}_X = (Y_{iX} : i = 1, 2, \ldots)$, $X \in \Phi$ are independent vectors of i.i.d. elements of $\mathbb{R}^d$, with Y_{iX}'s having the conditional distribution $\mathbf{P}(Y_{iX} \in \cdot \mid \Phi) = \mathcal{X}(X,\cdot)$,
- the random elements N_X, Y_X are independent given Φ, for all $X \in \Phi$.

Perturbation of a point processes; cont'd

$\mathbf{\Phi}^{pert}$ can be seen as independently replicating and translating points from the pp $\mathbf{\Phi}$, with replication kernel $\boldsymbol{\mathcal{N}}$ and the translation kernel $\boldsymbol{\mathcal{X}}$.



Perturbation of a point processes; cont'd

Perturbation of Φ is dcx monotone with respect to the replication kernel.

Proposition. [BB-Yogesh'11] *Consider a pp* Φ *with locally finite mean measure* $\alpha(\cdot)$ *and its two perturbations* Φ_j^{pert} $j = 1, 2$ *with the same translation kernel* $\mathcal{X}$ *and replication kernels* $\mathcal{N}_j$*,* $j = 1, 2$*, respectively. If* $\mathcal{N}(x, \cdot) \leq_{cx} \mathcal{N}(x, \cdot)$ *(convex ordering of the number of replicas; test functions* $\mathcal{F}$ *are all convex functions on* $\mathbb{R}$*) for* α*-almost all* $x \in \mathbb{R}^d$*, then* $\Phi_1^{pert} \leq_{dcx} \Phi_2^{pert}$*.*

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Proof. Using dcx comparison of some shot-noise fields.

Perturbed Poisson pp

Assume:

$\mathbf{\Phi}$ — (possibly inhomogeneous) Poisson pp,

arbitrary translation kernel,

$\boldsymbol{\mathcal{N}_1(x,\cdot)}$ Dirac measure on $\mathbb{Z}^+$ concentrated at 1,

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Poisson pp　　　　perturbed Poisson pp

Indeed, by Jensen's inequality $\mathcal{N}_1 \le_{cx} \mathcal{N}_2$.

Perturbed lattices

Assume:

$\boldsymbol{\Phi}$ — deterministic lattice,

(say uniform) translation kernel inside lattice cell,

and replication kernels:

$\boldsymbol{\mathcal{N}_0(x,\cdot) = Poi(1)}$,

$\boldsymbol{\mathcal{N}_1(x,\cdot) \leq_c Poi(1)}$,

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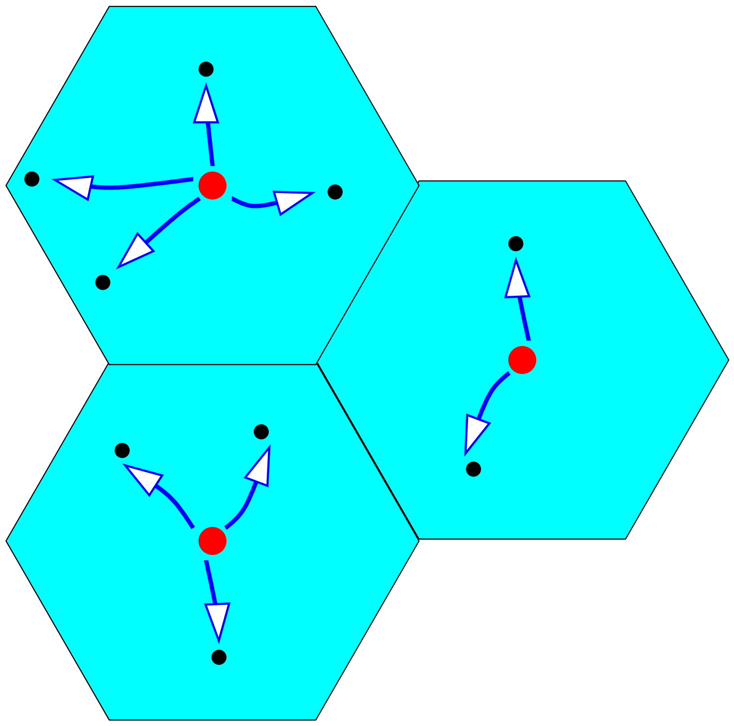

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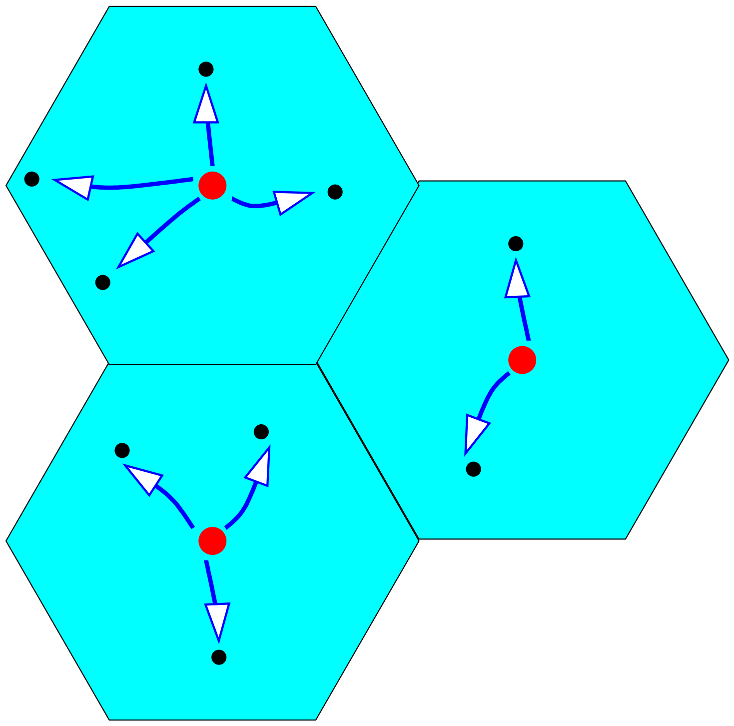

Then

$$\Phi_1^{pert} \leq_{dcx} \Phi_0^{pert} \leq_{dcx} \Phi_2^{pert}$$

sub-Poisson perturbed lattice

Poisson pp

super-Poisson perturbed lattice

Perturbed lattices; cont'd

cx ordered families of (discrete) random variables from smaller to larger:

- deterministic (constant);

Perturbed lattices; cont'd

cx ordered families of (discrete) random variables from smaller to larger:

- deterministic (constant);
- Hyer-Geometric $p_{HGeo(n,m,k)}(i) = \binom{m}{i}\binom{n-m}{k-i}/\binom{n}{k}$ $(\max(k-n+m,0) \leq i \leq m)$.

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Assuming parameters making equal means, we have
$const \le_{cx} HGeo \le_{cx} Bin \le_{cx} Poi \le_{cx} NBin \le_{cx} Geo$

Clustering and coverage

Capacity functional of Boolean Model

Let $C(\Phi, G)$ be Boolean model with germs Φ and with typical grain G.

Proposition. [BB-Yogesh'11] *If* $\Phi_1 \leq_{dcx} \Phi_2$ *then* $\mathbf{P}\left(C(\Phi_1, G) \cap B \neq \emptyset\right) \geq \mathbf{P}\left(C(\Phi_2, G) \cap B \neq \emptyset\right)$ *for all bBs* B *provided* G *is fixed (deterministic) compact grain or* Φ_i *are simple and have locally finite moment measures.*

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Proof. Void probabilities (complement of the capacity functional) can be expressed using the distribution function an extrema shot-noise:

$$\mathsf{P}\left(\Phi(B) = 0\right) = \mathsf{P}\left(\max_{X \in \Phi} 1(X \in B) \leq 0\right).$$

Coverage in SINR models

- Shot-noise field $\equiv$ interference field.

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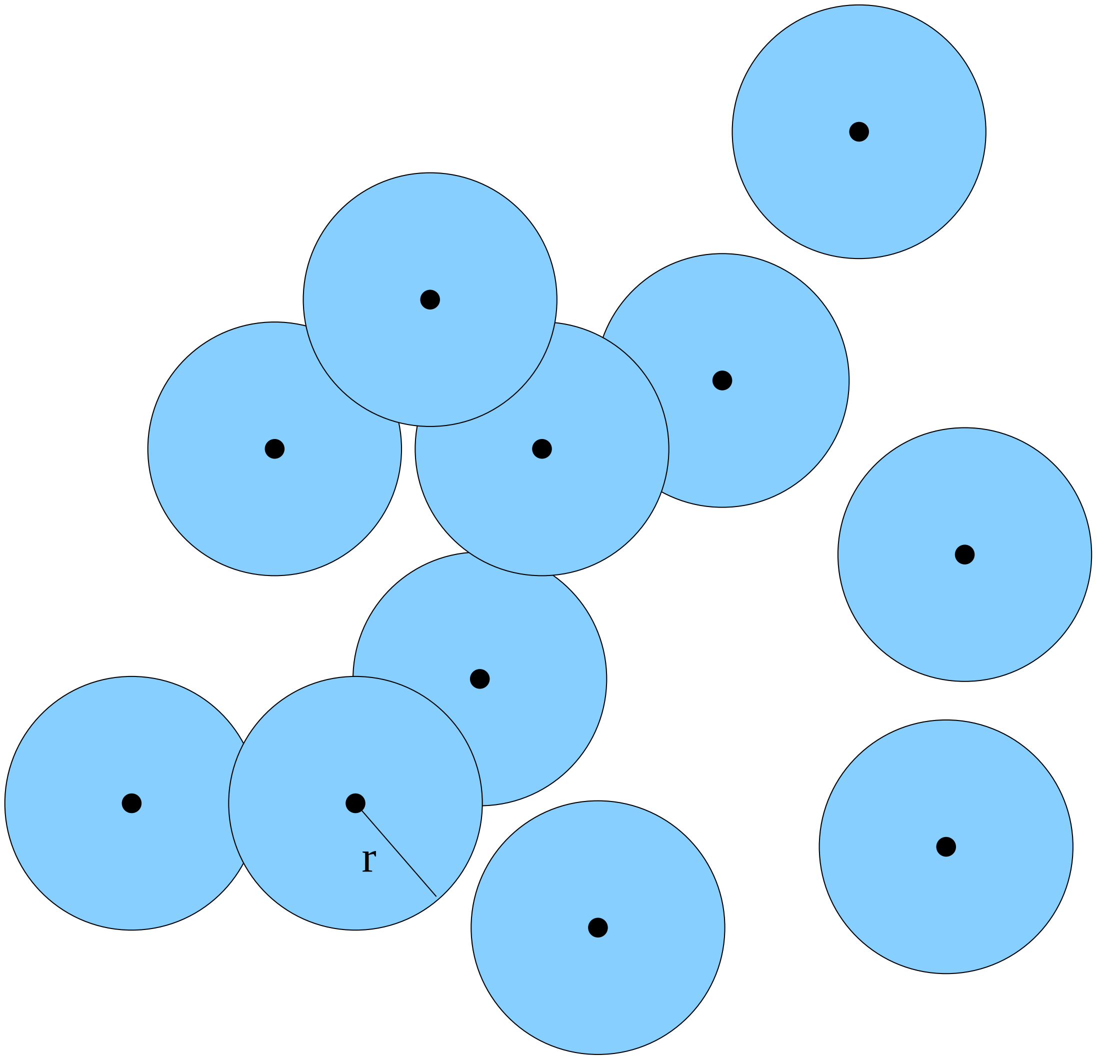
- Shot-noise field $\equiv$ interference field.
- Using the fact that dcx ordering of pp implies dcx ordering of the respective shot-noise fields we conclude that mean characteristics of models which are convex in interference I are increased(!) by the clustering of the pattern of interferers.
  Examples:
  - SINR coverage probability $\mathsf{P}\{S/(w+I) \geq const\}$ for signal power S with convex tail distribution function (Rayleigh fading case).
  - Shannon throughput $\mathsf{E}[\log(1+S/(w+I))]$.

Clustering and percolation

Continuum percolation

Boolean model $C(\Phi, r)$:
germs in Φ,
spherical grains of given radius r.



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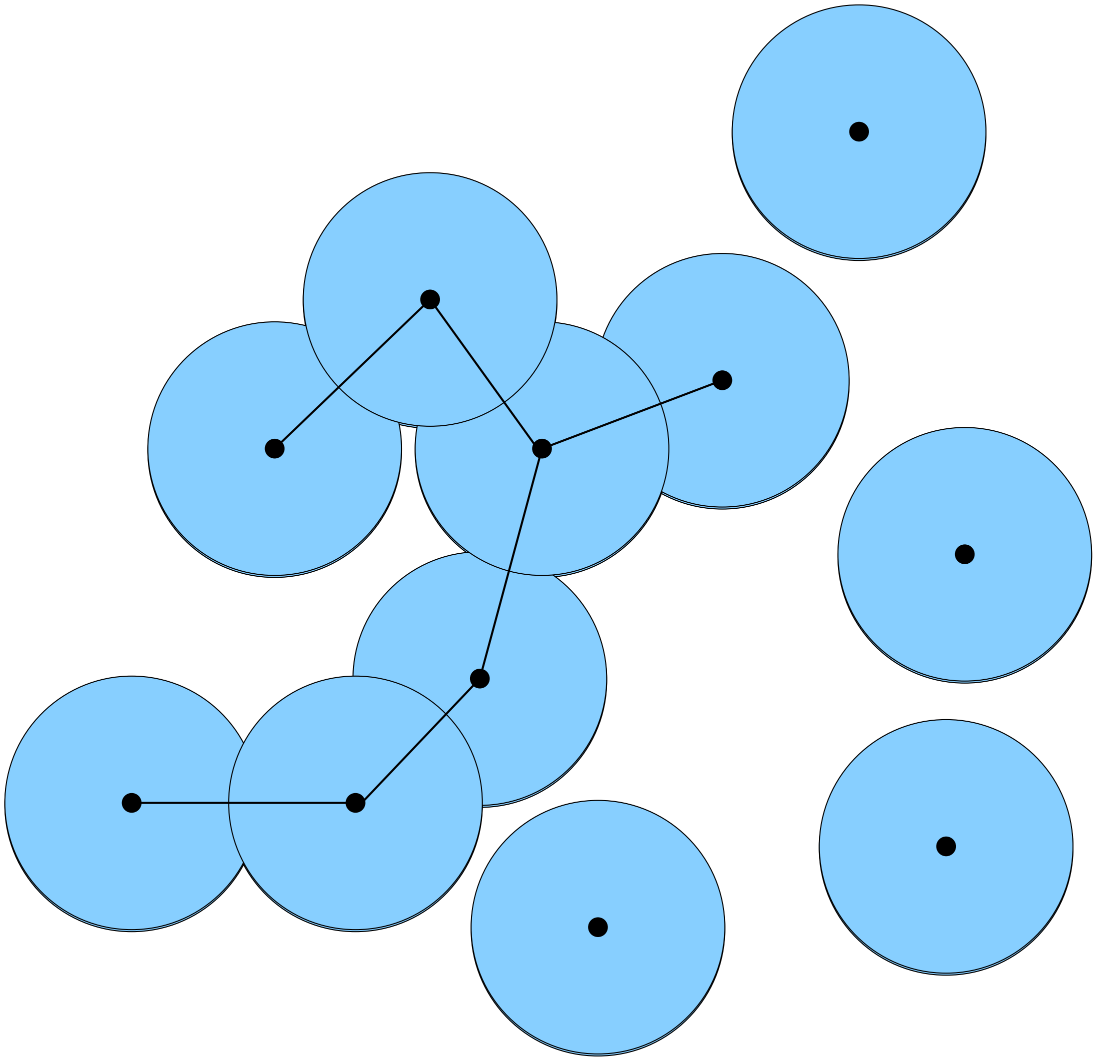

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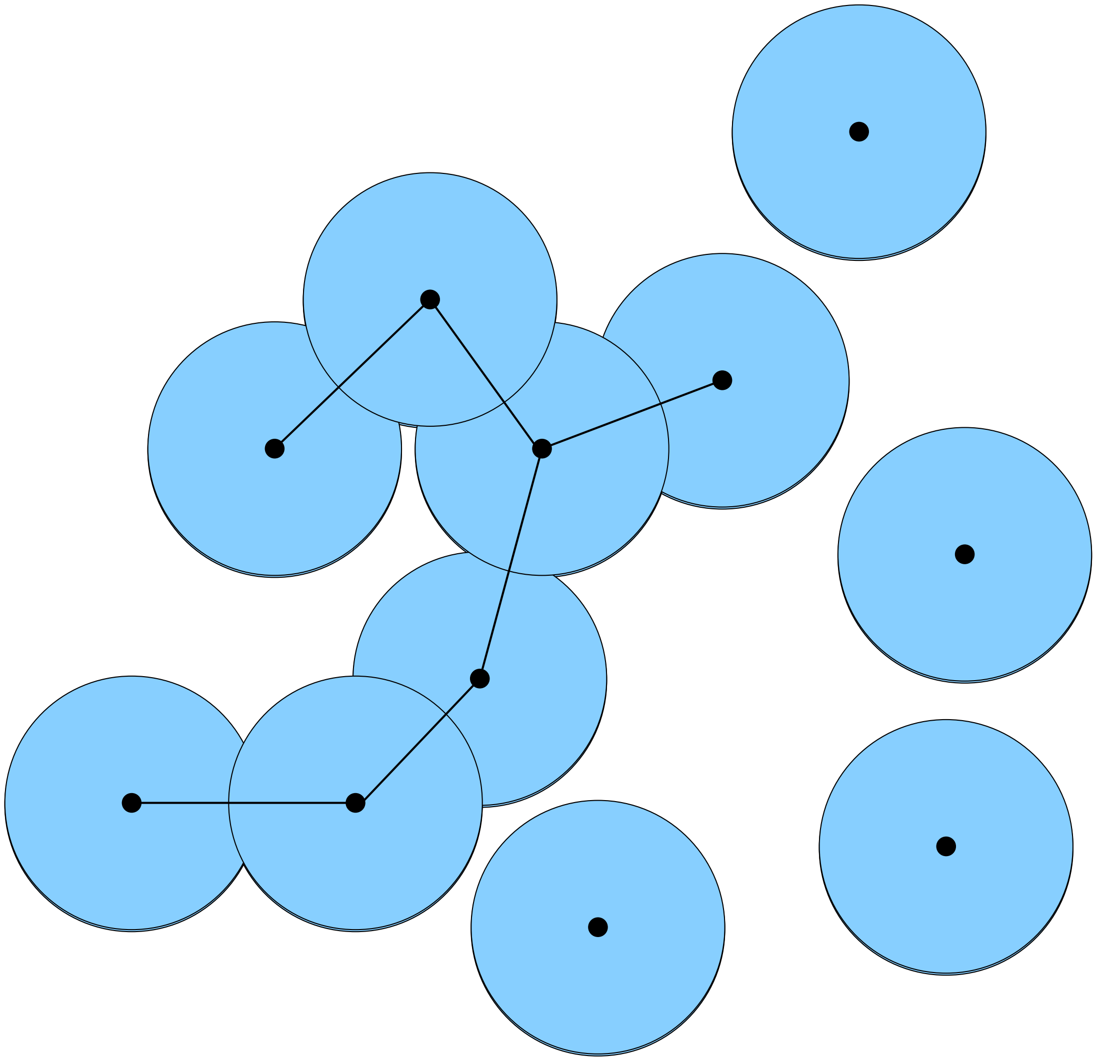

percolation $\equiv$ existence of an infinite connected subset (component).

Critical radius for percolation

Critical radius for the percolation in the Boolean Model with germs in Φ

$$r_c(\Phi) = \inf\{r > 0 : \mathsf{P}(C(\Phi, r)\text{percolates}) > 0\}$$

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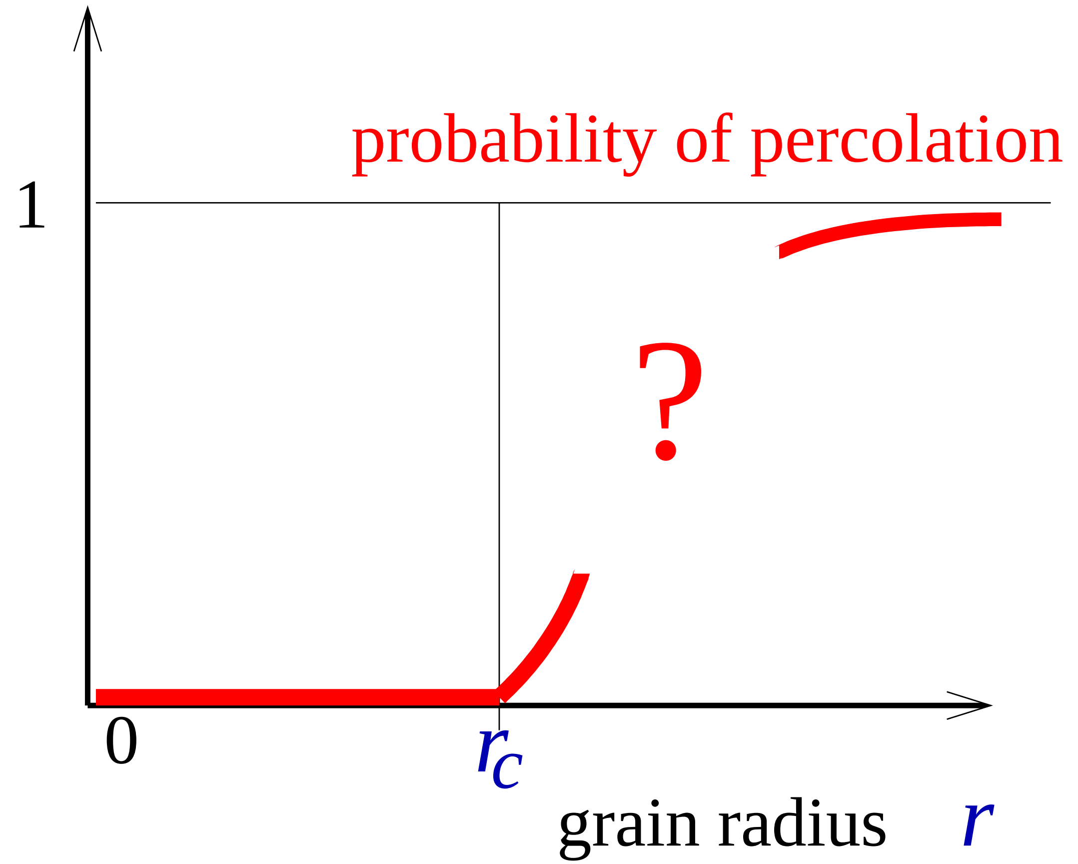
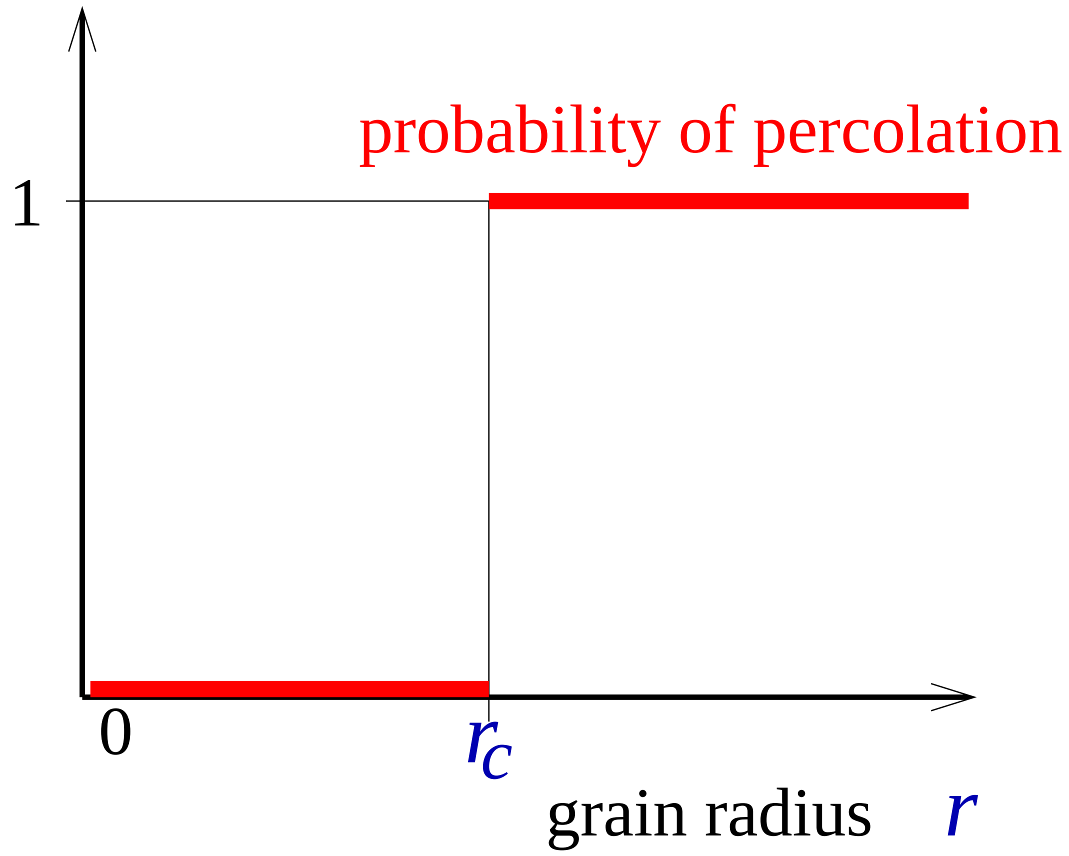


Phase transition in ergodic case

In the case when Φ is stationary and ergodic



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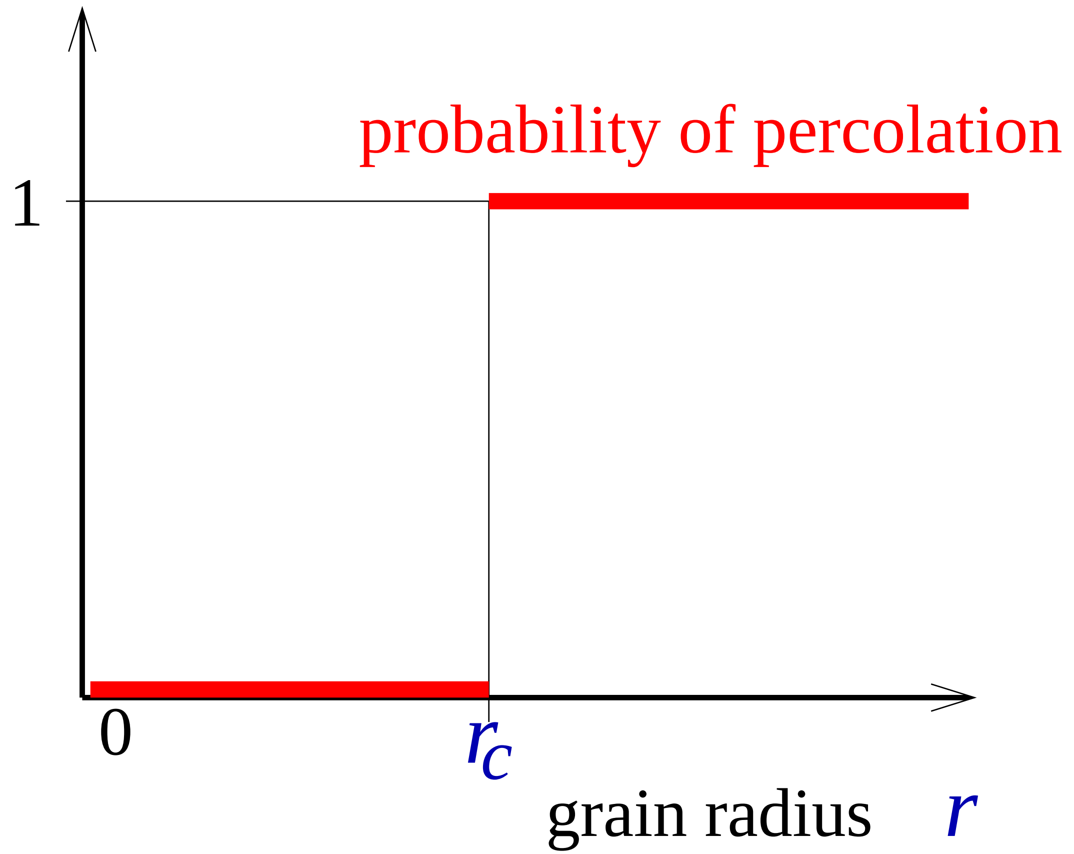

If $0 < r_c < \infty$ we say that the phase transition is non-trivial.

Clustering and percolation; Heuristic

Clustering worsens percolation.

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Point processes exhibiting more clustering of points should have larger critical radius r_c for the percolation of their continuum percolation models.

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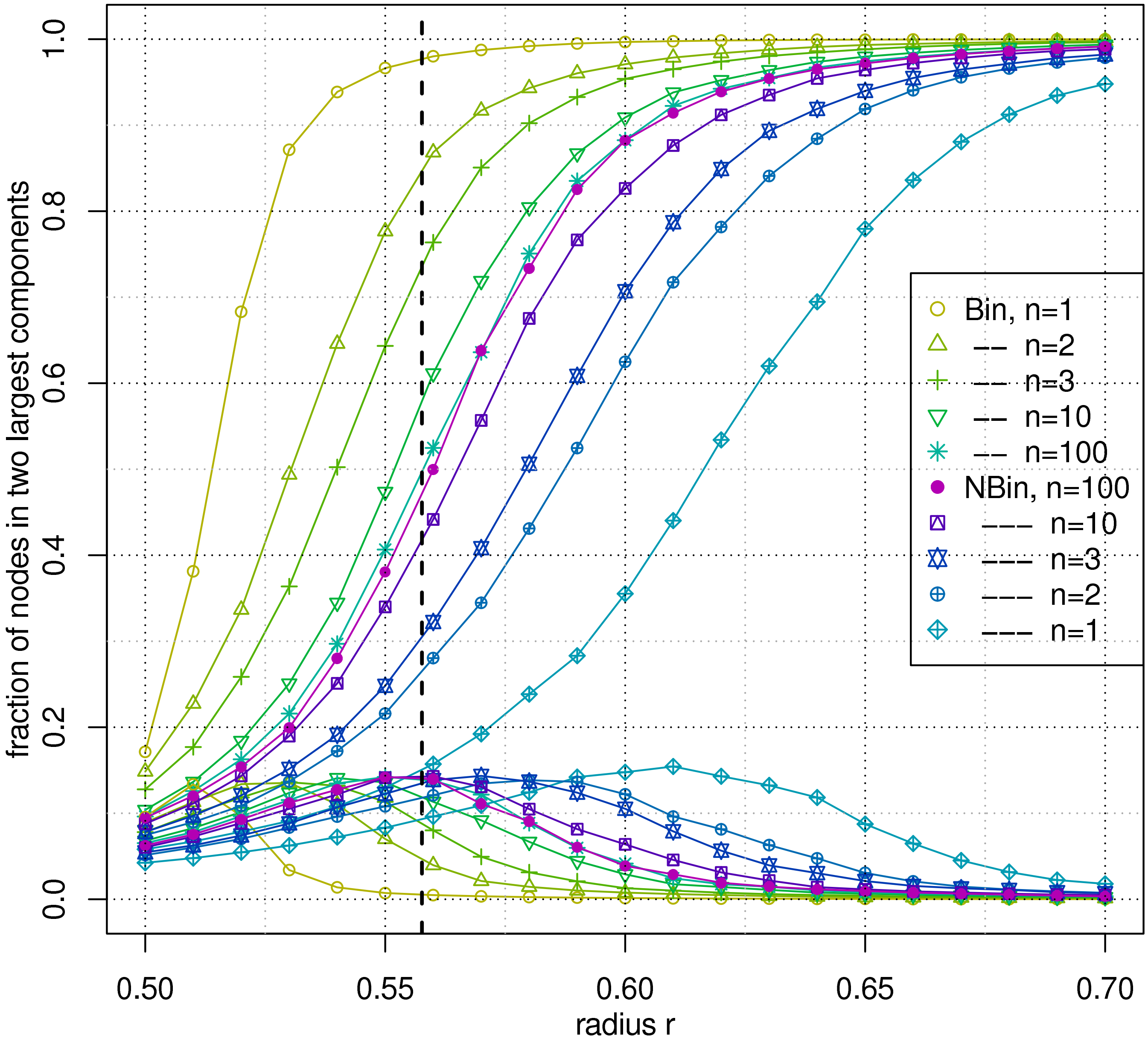
$$\Phi_1 \text{ “clusters less than” } \Phi_2 \quad \Rightarrow \quad r_c(\Phi_1) \leq r_c(\Phi_2).$$

Indeed, points lying in the same cluster of will be connected by edges for some smaller r *but points in different clusters need a relatively higher* r *for having edges between them, and percolation cannot be achieved without edges between some points of different clusters. Spreading points from clusters of ”more homogeneously” in the space should result in a decrease of the radius r for which the percolation takes place.*

Conjecture for perturbed lattices

$$\Phi_1 \quad \leq_{dcx} \quad \Phi_2$$
$$\Downarrow$$
$$r_c(\Phi_1) \quad \leq \quad r_c(\Phi_2)$$

$Bin(1,1) \quad =$
$const$
$Bin(1,1/n) \nearrow_{cx}$
$Poi(1)$
$NBin(n,1/(1+$
$n)) \searrow_{cx} Poi(1)$
$NBin(1,1/2) =$
$Geo(1/2)$



Counterexample

One can construct Poisson-Poisson cluster pp of any intensity $a > 0$ and $r_c = 0$!

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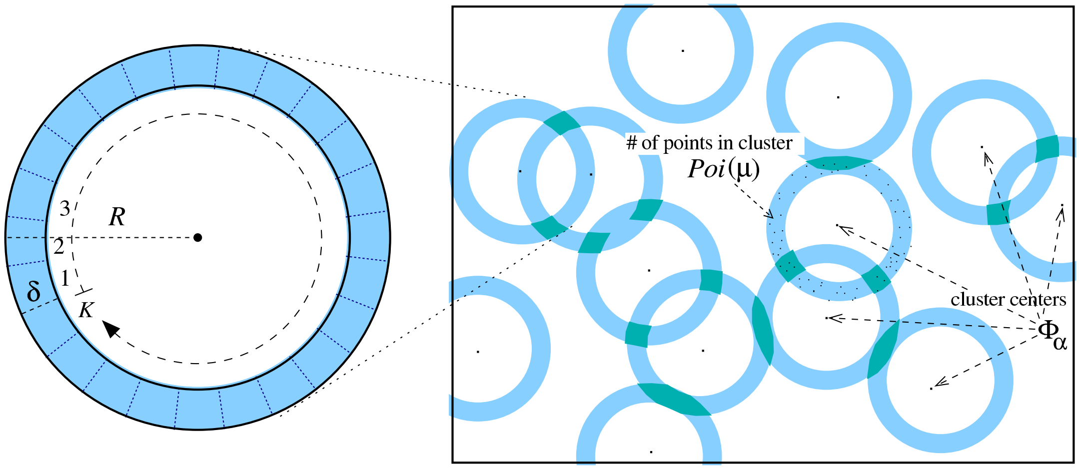
Poisson-Poisson cluster pp $\Phi_\alpha^{R,\delta,\mu}$ with annular clusters

Φ_α — Poisson (parent) pp of intensity α on $\mathbb{R}^2$, Poisson clusters of total intensity μ, supported on annuli of radii $R - \delta, R$.



We have $\Phi_\lambda \leq_{dcx} \Phi_\alpha^{R,\delta,\mu}$, where Φ_λ is homogeneous Poisson pp of intensity $\lambda = \alpha\mu$.

Proposition. [BB-Yogesh'11] *Given arbitrarily small $a, r > 0$, there exist constants α, μ, δ, R such that $0 < \alpha, \mu, \delta, R < \infty$, the intensity $\alpha\mu$ of $\Phi_\alpha^{R,\delta,\mu}$ is equal to a and the critical radius for percolation $r_c(\Phi_\alpha^{R,\delta,\mu}) \leq r$.*

Phase transitions for sub-Poisson pp

Proposition. [BB-Yogesh'11]*Let* Φ *be a stationary pp on* $\mathbb{R}^d$*, weakly sub-Poisson (void probabilities and moment measures smaller than for the Poisson pp of some intensity* λ*). Then*

$$0 < \frac{1}{(2^d\lambda(3^d-1))^{1/d}} \leq r_c(\Phi) \leq \frac{\sqrt{d}(\log(3^d-2))^{1/d}}{\lambda^{1/d}} < \infty.$$

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Similar results for

- k-percolation (percolation of k-covered subset) for dcx sub-Poisson.
- word percolation,
- SINR-graph percolation (graph on a shot-noise germ-grain model).

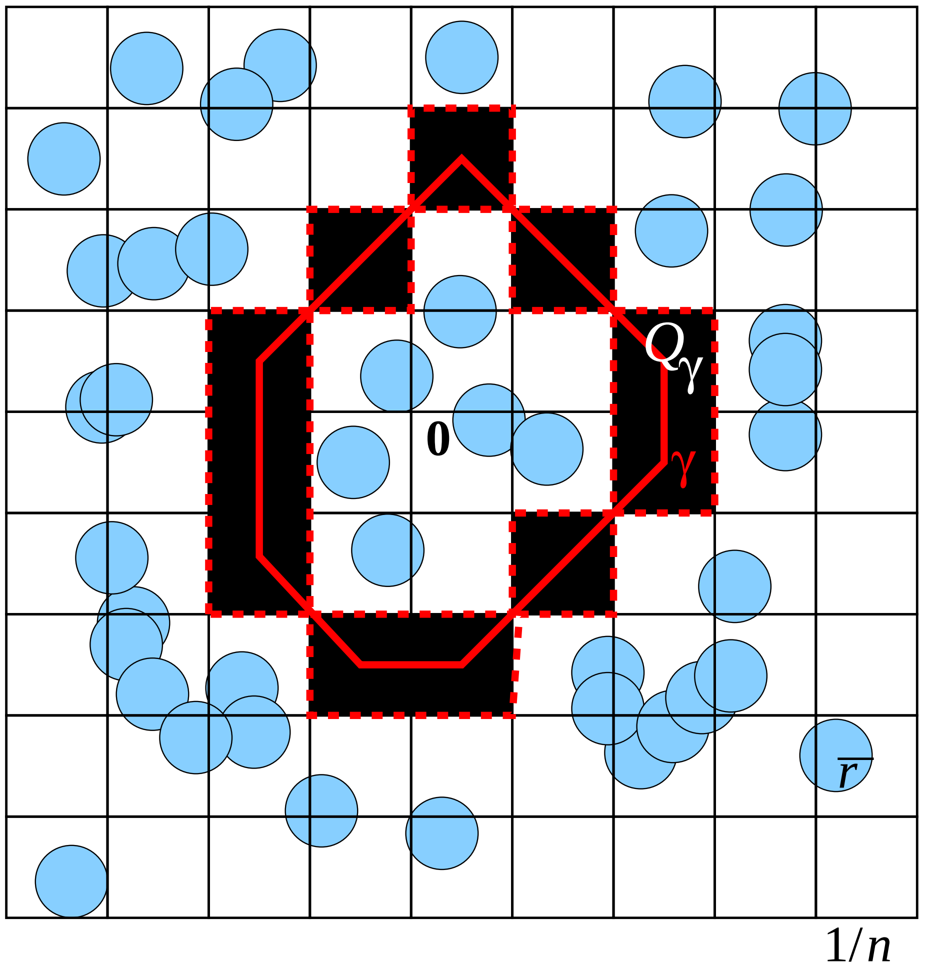
Hint 1: An “upper” critical radius

Define a new critical radius

$$\bar{r}_c = \inf\left\{r > 0 : \forall n \geq 1, \sum_{\gamma \in \Gamma_n} \mathbf{P}\left(C(\Phi, r) \cap Q_\gamma = \emptyset\right) < \infty\right\}.$$

By Peierls argument

$$r_c(\Phi) \leq \bar{r}_c(\Phi).$$



Peierls argument

- A sufficient condition for percolation: the maximal number of closed (not tuching the Boolean Model), disjoint contours around the origin is finite.

Peierls argument

- A sufficient condition for percolation: the maximal number of closed (not tuching the Boolean Model), disjoint contours around the origin is finite.
- Even stronger condition: expected number of such closed contours is finite.

$$
\begin{aligned}
&\mathbf{E}(\text{number of closed contours}) \\
&= \mathbf{E}\left(\sum_{\gamma\in\Gamma_n} 1(\text{contour } \gamma \text{ is closed})\right) \\
&= \sum_{\gamma\in\Gamma_n} \mathbf{P}\,(\text{contour } \gamma \text{ is closed}) \\
&= \sum_{\gamma\in\Gamma_n} \mathbf{P}\,(C(\Phi, r) \cap Q_\gamma = \emptyset) < \infty\,.
\end{aligned}
$$

“Upper” critical radius; cont’d

Fact. *If* $\mathbf{\Phi_1} \leq_{dcx} \mathbf{\Phi_2}$ *then* $\overline{r}_c(\mathbf{\Phi_1}) \leq \overline{r}_c(\mathbf{\Phi_2})$.

“Upper” critical radius; cont’d

Fact. *If* $\boldsymbol{\Phi_1 \le_{dcx} \Phi_2}$ *then* $\boldsymbol{\overline{r}_c(\Phi_1) \le \overline{r}_c(\Phi_2)}$.
Ordering of void probabilities of $\boldsymbol{\Phi_i}$ is enough for RGG.
$\boldsymbol{dcx}$ needed for Boolean models with arbitrary grain.

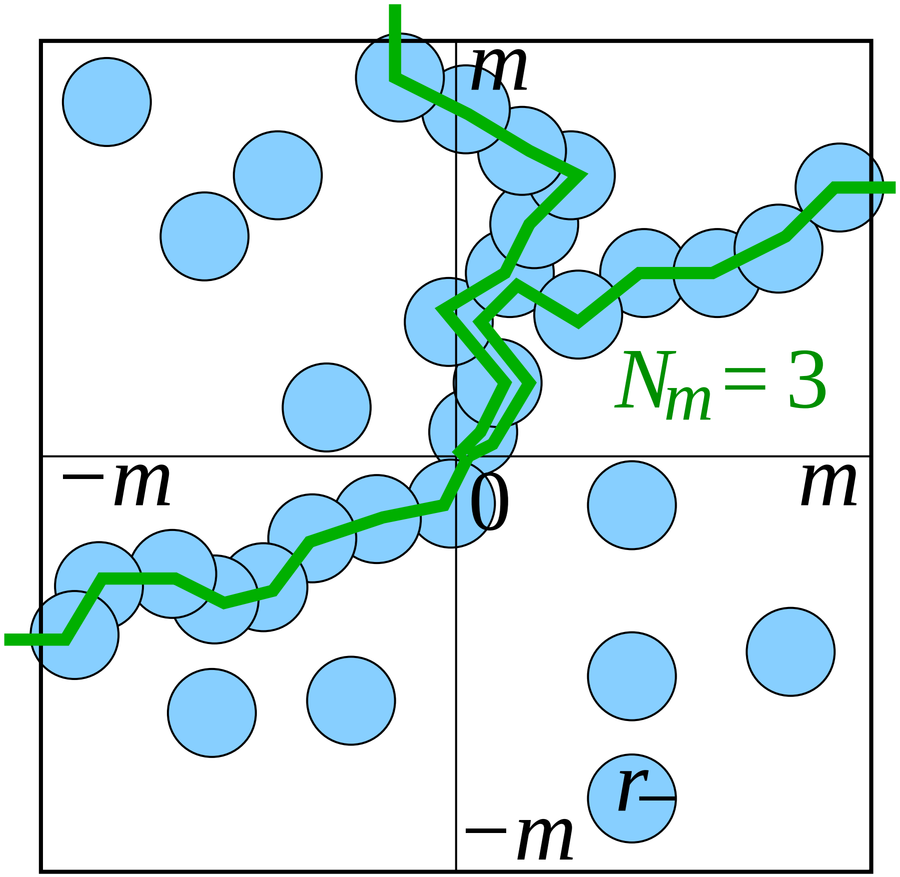
Hint 2: A "lower" critical radius

Define a new critical radius

$$\underline{r}_c(\Phi) := \inf\left\{r > 0 : \liminf_{m\to\infty} \mathbf{E}(N_m(\Phi, r)) > 0\right\}.$$

By Markov inequality

$$\underline{r}_c(\Phi) \leq r_c(\Phi).$$



“Lower” critical radius; cont’d

Fact. *If* $\mathbf{\Phi_1} \le_{dcx} \mathbf{\Phi_2}$ *then* $\underline{r}_c(\mathbf{\Phi_1}) \ge \underline{r}_c(\mathbf{\Phi_2})$.

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Inequality reversed! In clustering pp, whenever there is at least one path of some given length, there might be actually so many such paths, that the inequality for the expected numbers of paths are reversed.

“Lower” critical radius; cont’d

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Ordering of moment measures of $\mathbf{\Phi}_i$ is enough for RGG.

Sandwich inequality for the critical radii

Corollary. *If* $\Phi_1 \leq_{dcx} \Phi_2$ *then*

$$\underline{r}_c(\Phi_2) \leq \underline{r}_c(\Phi_1) \leq r_c(\Phi_1) \leq \overline{r}_c(\Phi_1) \leq \overline{r}_c(\Phi_2).$$

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Double phase transition for Φ_2

$$0 < \underline{r}_c(\Phi_2) \le \overline{r}_c(\Phi_2) < \infty$$

$$\Downarrow$$

usual phase transition for all $\Phi_1 \le_{dcx} \Phi_2$

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The double phase transition holds for Poisson pp and thus ensures the usual phase transition of all sub-Poisson pp.

Clustering and first passage percolation

Routing on random space-time SINR graph

Result.[Baccelli-BB-Misradeghi (2011)] *Existence of stochastically too large voids in Poisson pp is the reason of infinite end-to-end packet-delivery delays in a time-space SINR model, studied in the framework of a first passage percolation problem.*
The same problem studied on some less clustering pp (that can be shown dcx sub Poisson) gives finite delays.

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thank you