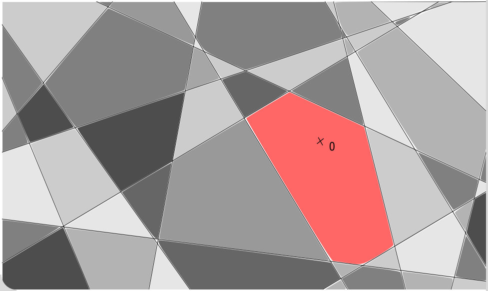
ON THE ZERO CELL OF A PARAMETRIC CLASS OF POISSON HYPERPLANE TESSELLATIONS

Daniel Hug | March 29, 2012

INSTITUTE OF STOCHASTICS, KIT



Plan

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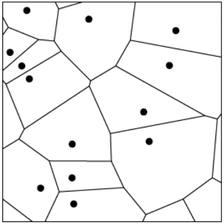
1. From Poisson Voronoi tessellations to a **parametric class** of Poisson hyperplane tessellations
2. **Formulas and numerical calculations** for expectation and variance of the volume of the zero cell
3. **Asymptotic behaviour** as the dimension goes to infinity for different choices of parameters
4. **Application** to the slicing problem

Stationary PV tessellations



- A stationary Poisson point process Y in $\mathbb{R}^n$ generates a stationary Poisson Voronoi tessellation.
- The tessellation is completely determined by the intensity $\lambda = \mathbb{E}[\text{card}(Y \cap [0,1]^n)]$.
- For the typical cell C_t of a Poisson Voronoi tessellation we have: $C_t = C_0(Y \cup \{0\})$

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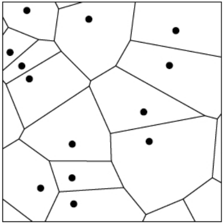

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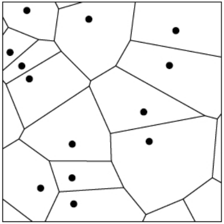

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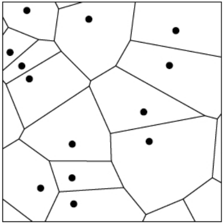

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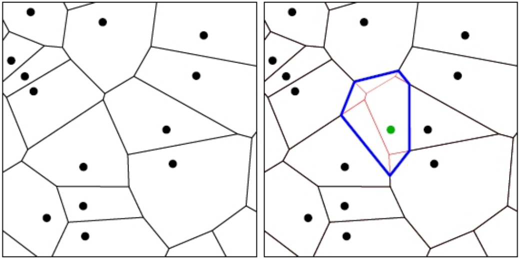

From a forthcoming lecture note on stochastic geometry edited by E. Spodarev

Shape of large C_t: Kendall's problem

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H, Reitzner & Schneider (2004)

Let C_t be the typical cell of a PVT derived from a stationary PPP Y with intensity λ in $\mathbb{R}^n$. There is a constant $c_0 = c_0(n)$ such that:

If $\epsilon \in (0, 1)$ and $a \geq 1$, then

$$\mathbb{P}\{\vartheta(C_t) \leq \epsilon \mid V_n(C_t) \geq a\} \geq 1 - c \exp\left\{-c_0 \epsilon^{(n+3)/2} a^{n/k} \lambda\right\}$$

where $c = c(n, \epsilon)$.

Related contributions:

- Kovalenko
- Calka, Calka & Schreiber
- H & Schneider

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Random mosaics in high dimensions

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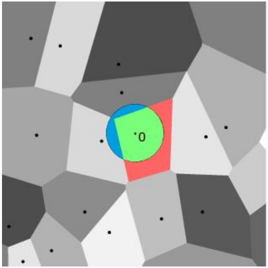
- So far random tessellations and associated random polytopes have been considered mainly in arbitrary but fixed dimensions.
- Instead of the dimension, the asymptotics of other parameters were studied (e.g., a bound a for the volume is introduced and $a \to \infty$).
- Quite recently, an interesting asymptotic result (as $n \to \infty$) has been investigated for typical cells of Poisson Voronoi tessellations.

Volume and shape of the typical cell



The map $u \mapsto V_n(C_t \cap B_u^n)$ contains information about the shape of C_t

B_u^n: full-dimensional ball with center o and n-dimensional volume u

V_n: n-dimensional volume

Alishahi & Sharifitabar '08

$\mathbb{E}[V_n(C_t)] = \frac{1}{\lambda}$, for all dimensions n

$\mathrm{Var}[V_n(C_t)] \to 0$, for $n \to \infty$ (exponentially fast)

$\mathbb{E}[V_n(C_t \cap B_u^n)] = \frac{1}{\lambda}(1 - e^{-\lambda u})$, for $u \in (0, \infty)$ and all dimensions n

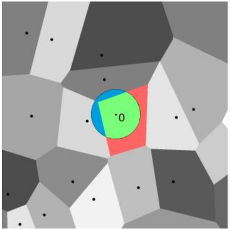
$\mathrm{Var}(V_n(C_t \cap B_u^n)) \to 0$, for $n \to \infty$ (exponentially fast)

Hence, $V_n(C_t)$ and $V_n(C_t \cap B_u^n)$ converge in squared mean for $n \to \infty$

Newman & Rinott '85

A consequence:

$$\Delta_s(C_t, B_u^n) := V_n(C_t \setminus B_u^n) + V_n(B_u^n \setminus C_t)$$

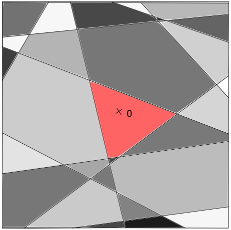


$$\mathbb{E}\Delta_s(C_t, B_u^n) \geq \ln(2)/\lambda$$

and

$$\Delta_s(C_t, B_u^n) - \mathbb{E}\Delta_s(C_t, B_u^n) \to 0 \quad \text{in } L^2 \text{ as } n \to \infty.$$

Hyperplane tessellations



- A Poisson hyperplane process X generates a Poisson hyperplane tessellation *(no stationarity is assumed!)*.
- X is completely determined by its intensity measure $\Theta : \mathcal{B}(A(n, n-1)) \to [0, \infty], \quad \mathcal{A} \mapsto \mathbb{E}[\mathrm{card}(X \cap \mathcal{A})]$.
- The zero cell Z_0 is the cell containing the origin.

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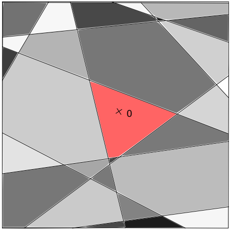

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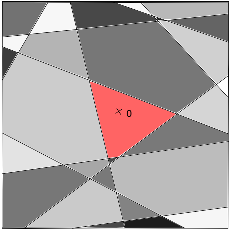

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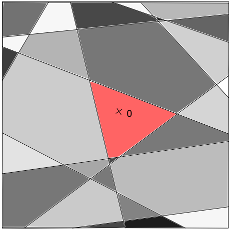

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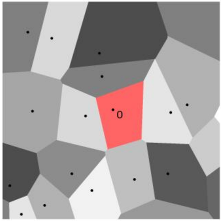
Connection between C_t and Z_0

- $C_t = Z_0$, where Z_0 is the zero cell of a *special* hp X.
- X is the process of mid-hyperplanes of o and the points of Y.
- X is the unique Poisson hyperplane process with intensity measure

$$\Theta(\mathcal{A}) = \frac{2\gamma_{Voronoi}}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbb{1}_{\mathcal{A}}(H(u,t))t^{n-1} dt\, \mathcal{H}^{n-1}(du),$$

for $\mathcal{A} \in \mathcal{B}(A(n, n-1))$,

where $\gamma_{Voronoi} := n\kappa_n 2^{n-1}\lambda$ depending on the dimension n.



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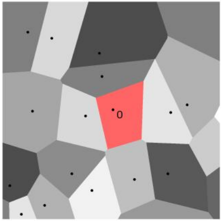

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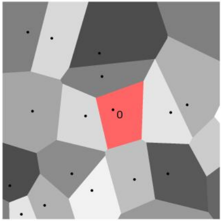

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A parametric class

Kendall's problem: H & Schneider '07

Let X be a Poisson hp with an intensity measure Θ of the form

$$\Theta(\mathcal{A}) = \frac{2\gamma}{n\kappa_n} \int_{S^{n-1}} \int_0^\infty \mathbb{1}_{\mathcal{A}}(H(u,t)) t^{r-1} dt\, \mathcal{H}^{n-1}(du),$$

$\mathcal{A} \in \mathcal{B}(A(n,n-1))$, with intensity $\gamma > 0$, distance exponent $r \in (0,\infty)$.

- Such a hp is said to be of type **DI** (**D**istance exponent **I**ntensity).
- X is always isotropic, but stationary only for $r = 1$.
- For $r = 1$ the hp X is the unique stationary, isotropic hp with intensity γ.
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Motivation

- The hpes of type DI are completely determined by $r, \gamma > 0$.
- **Idea:** Explore the type of questions considered by Alishahi and Sharifitabar for the zero cell Z_0 generated by the hpes of type DI.
- How does the choice of r and γ influence the (asymptotic) properties of Z_0?
- Z_0 provides an interesting model of a random polytope in high dimensions.

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Volume of the zero cell

Proposition (Hörrmann & H)

For $k \in \mathbb{N}$, we have

$$\Gamma\left(\frac{n}{r}+1\right)^{k} \kappa_n^k \left(\frac{n\kappa_n r}{2\gamma c(n,r)}\right)^{\frac{kn}{r}}$$

$$\leq \mathbb{E}[V_n(Z_0)^k]$$

$$\leq \Gamma\left(\frac{kn}{r}+1\right) \kappa_n^k \left(\frac{n\kappa_n r}{2\gamma c(n,r)}\right)^{\frac{kn}{r}}.$$

In particular, for $k = 1$ we get

$$\mathbb{E}[V_n(Z_0)] = \Gamma\left(\frac{n}{r}+1\right) \kappa_n \left(\frac{n\kappa_n r}{2\gamma c(n,r)}\right)^{\frac{n}{r}}.$$

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Behaviour as $r \to \infty$

- For constant intensity $\gamma > 0$, we have

$$\lim_{r\to\infty} \mathbb{E}[V_n(Z_0)^k] = \kappa_n^k \quad \text{for } k \in \mathbb{N}.$$

- For $\gamma = \frac{n\kappa_n r}{2c(n,r)} \left(\Gamma(\frac{n}{r}+1)\kappa_n\right)^{\frac{r}{n}}$, we have

$$\mathbb{E}[V_n(Z_0)] = 1$$

and

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Theorem (Hörrmann & H)

$$\mathrm{Var}[V_n(Z_0)] = \frac{8\pi b_{n,2}}{r}\,\Gamma\left(\frac{2n}{r}\right)\left(\frac{n\kappa_n r}{2\gamma c(n,r)}\right)^{\frac{2n}{r}}$$

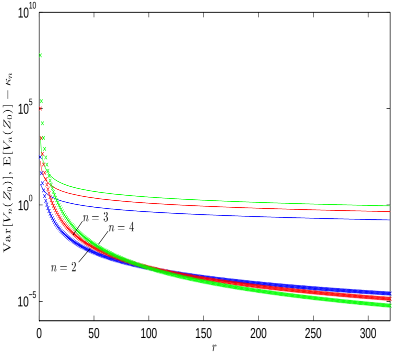
$$\times \int_0^{\pi}\int_0^{1}\left(\frac{1}{F_r(t,\varphi)^{\frac{2n}{r}}} - \frac{1}{(t^r+1)^{\frac{2n}{r}}}\right) t^{n-1}(\sin\varphi)^{n-2}\,dt\,d\varphi,$$

where

$$F_r(t,\varphi) := \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})}\left(t^r \int_{-\frac{\pi}{2}}^{\alpha(\varphi,t)} (\cos\theta)^r\,d\theta + \int_{\alpha(\varphi,t)-\varphi}^{\frac{\pi}{2}} (\cos\theta)^r\,d\theta\right)$$

for $(t,\varphi) \in [0,1] \times (0,\pi)$.

Numerical calculations for large r



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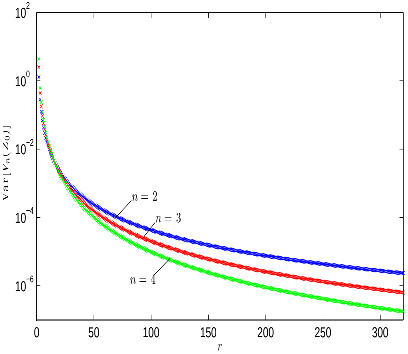
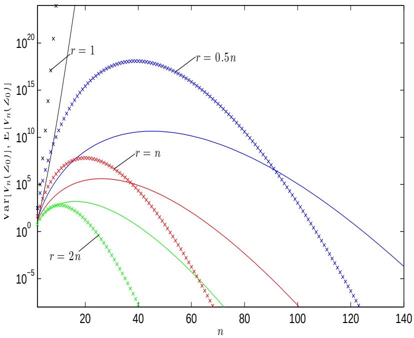
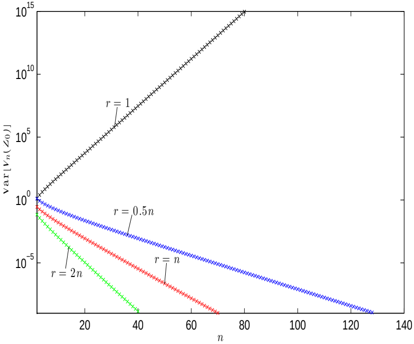


$\gamma(n, r)$ s.t. $\mathbb{E}[V_n(Z_0)] = 1$

Numerical calculations for large n





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Theorem (Hörrmann & H)

$$D(n,r)E(n,r) \;\le\; Var[V_n(Z_0)] \;\le\; D(n,r)E(n,r)4^{\frac{2n}{r}+1},$$

where

$$D(n,r) := \frac{n\kappa_n^2}{r}\Gamma\left(\frac{2n}{r}+1\right)\left(\frac{n\kappa_n r}{4\gamma c(n,r)}\right)^{\frac{2n}{r}}$$

and

$$E(n,r) := \frac{n\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n-1}{2}\right)}\int_0^{\pi}(\sin\varphi)^{n-2}\int_0^1\Big[t^{n+r-1}M(\alpha(\varphi,t),r)$$
$$+\,t^{n-1}M(\varphi-\alpha(\varphi,t),r)\Big]\,dt\,d\varphi$$

with

$$M(v,r) := \frac{\Gamma\left(\frac{r}{2}+1\right)}{\sqrt{\pi}\Gamma\left(\frac{r+1}{2}\right)}\int^{\frac{\pi}{2}}(\cos\theta)^r d\theta, \quad v\in[-\frac{\pi}{2},\frac{\pi}{2}].$$

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$$\left. + t^{n-1}M(\varphi-\alpha(\varphi,t),r)\right] dt\, d\varphi$$

with

$$M(v,r) := \frac{\Gamma(\frac{r}{2}+1)}{\sqrt{\pi}\Gamma(\frac{r+1}{2})}\int^{\frac{\pi}{2}} (\cos\theta)^r d\theta, \quad v\in[-\frac{\pi}{2},\frac{\pi}{2}].$$

Theorem (Hörrmann & H)

$$D(n,r)E(n,r) \;\le\; Var[V_n(Z_0)] \;\le\; D(n,r)E(n,r)4^{\frac{2n}{r}+1},$$

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Behaviour as $r \to \infty$

For constant intensity $\gamma > 0$ we have

$$\lim_{r\to\infty} \mathbb{E}[V_n(Z_0)^k] = \kappa_n^k \quad \text{for } k \in \mathbb{N},$$

$$\mathrm{Var}[V_n(Z_0)] = O\left(\frac{1}{r}\right).$$

For $\gamma = \frac{n\kappa_n r}{2c(n,r)} \left(\Gamma(\frac{n}{r}+1)\kappa_n\right)^{\frac{r}{n}}$, we have

$$\mathbb{E}[V_n(Z_0)] = 1,$$

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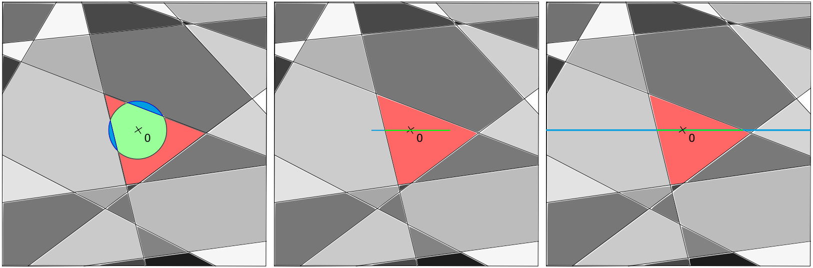
Behaviour of $V_n(Z_0)$ as $n \to \infty$

The previous proposition and theorem imply

constant r, constant γ	constant r, $\gamma(r, n)$	$r = an, a > 0$, constant γ	$r = an, a > 0$, $\gamma(a, n)$ **DI***
$\mathbb{E} \to \infty$ $\mathrm{Var} \to \infty$	$\mathbb{E} = 1$ $\mathrm{Var} \to \infty$	$\mathbb{E} \to 0$ $\mathrm{Var} \to 0$	$\mathbb{E} = 1$ $\mathrm{Var} \to 0$

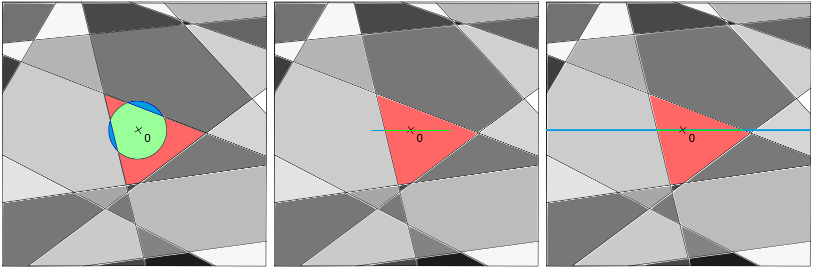
Further studied properties



Volume of section of Z_0 with a

- full-dim. ball: $V_n(Z_0 \cap B_u^n)$, for $u \in (0, \infty)$
- low-dim. ball: $V_m(Z_0 \cap B_u^m)$, for $u \in (0, \infty)$, $m \leq n$
- (hyper-)plane containing 0: $V_m(Z_0 \cap L)$, for $L \in G(n, m)$, $m \leq n$.

Further studied properties



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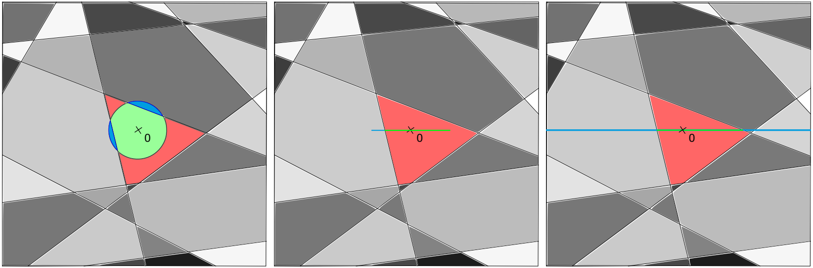

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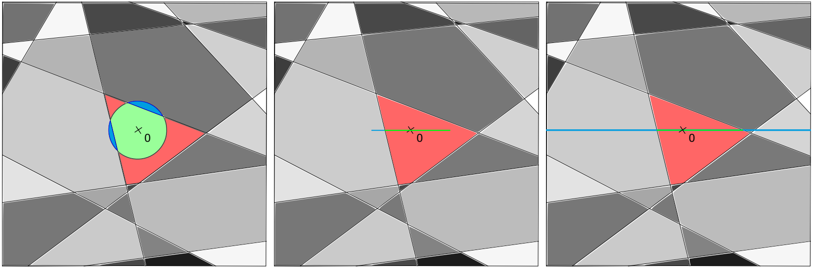

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r proportional to the dimension

	$V_n(Z_0)$	$V_m(Z_0 \cap B_u^m)$	$V_{n-l}(Z_0 \cap B_u^{n-l})$	$V_{n-l}(Z_0 \cap L)$
$r = an$, $a > 0$ $\gamma(a, n)$, (Typ **DI***)	$\mathbb{E} \to 1$ $\mathrm{Var} \to 0$	$\mathbb{E} \to u$ $\mathrm{Var} \to 0$	$\mathbb{E} \to Int(a, u, l)$ $\mathrm{Var} \to 0$	$\mathbb{E} \to e^{\frac{l}{2}}$ $\mathrm{Var} \to 0$

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$\gamma(1, n) = \gamma_{Voronoi}(n)$!

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The Voronoi case

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Slicing problem and isotropic constant

Let $K \subset \mathbb{R}^n$ be a convex body with $V_n(K) = 1$. Is there a hyperplane $H \subset \mathbb{R}^n$ such that

$$V_{n-1}(K \cap H) \geq c$$

with some universal constant $c > 0$?

For a convex body $K \subset \mathbb{R}^n$, the isotropic constant L_K of K is defined by

$$n \cdot L_K^2 := \min_T \frac{1}{V_n(TK)^{1+\frac{2}{n}}} \int_{TK} \|x\|^2 \, dx.$$

Is there a universal constant C such that

$$L_K \leq C$$

for all convex bodies $K \subset \mathbb{R}^n$ and all $n \in \mathbb{N}$?

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Some known results

- $L_K \leq c \cdot n^{1/4} \log(n)$ Bourgain (1991)
- $L_K \leq c \cdot n^{1/4}$ Klartag (2006)
- Conjecture holds for special classes of bodies (zonoids, ...)
- $L_P \leq C \cdot (f_0(P)/n)^{1/2}$ Alonso-Gutiérrez, Bastero, Bernués, Wolff (2010)
- The isotropic constant of certain classes of random polytopes is bounded with high probability:
  - Gaussian polytopes Klartag, Kozma (2008)
  - Random polytopes whose vertices have independent coordinates
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Sections through random polytopes

Theorem (Hörrmann & H)

Let $\overline{Z}_0 := \big(V_n(Z_0)\big)^{-\frac{1}{n}} Z_0$ be the normalized zero cell of a PHP in $\mathbb{R}^n$ with distance exponent $r = an$ and intensity $\gamma(a, n, \lambda)$. Then, for any $L \in G(n, n-1)$,

$$\mathbb{P}\left\{ V_{n-1}(\overline{Z}_0 \cap L) > \frac{\sqrt{e}}{2} \right\} \geq 1 - C\left(\frac{1}{\sqrt{n}} \left(\frac{2}{\sqrt{5}} \right)^{an} + \left(\frac{2}{\sqrt{5}} \right)^{n} \right)$$

for a universal constant $C > 0$.