The convex hull of randomly chosen points

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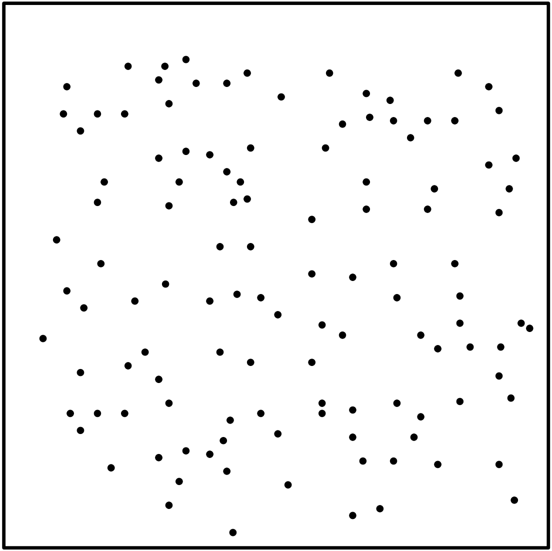
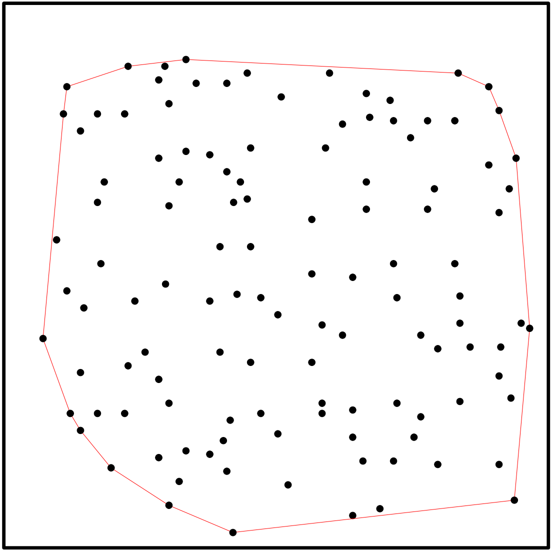




- Rényi and Sulanke (1963)

$$\mathbb{E}N_n(K) = \frac{2}{3} r \log n + O(1)$$

- Rényi and Sulanke (1964)

$$\mathbb{E}A_n(K) = 1 - \frac{2}{3} r \frac{\log n}{n} + O\left(\frac{1}{n}\right)$$

- Efron (1965)

$$\mathbb{E}N_{n+1}(K) = (n+1)\left(1 - \mathbb{E}A_n(K)\right)$$

$$\Rightarrow \mathbb{E}A_n(K) = 1 - \frac{\mathbb{E}N_{n+1}(K)}{n+1}$$

- B. (1984)

$$\mathbb{E}N_n(\text{triangle}) = 2\sum_{k=1}^{n-1}\frac{1}{k}$$

$$\mathbb{E}N_n(\text{square}) = \frac{8}{3}\left[\sum_{k=1}^{n}\frac{1}{k}\left(1-\frac{1}{2^k}\right)-\frac{1}{n\,2^n}\right]$$

- B. (1984)

$$
\begin{aligned}
\mathbb{E}N_n(\text{regular hexagon}) = 4 \Bigg[& \sum_{k=1}^{n} \frac{1}{k} \left(1 - \left(\frac{5}{6}\right)^k\right) + \\
& + 2\left(\left(\frac{1}{6}\right)^n + \left(\frac{5}{6}\right)^n - \left(\frac{1}{2}\right)^n\right) - 1 - \frac{1}{n\,6^n} + \\
& + 2\binom{n}{2}\frac{1}{6^n}\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\left((-1)^n\,7^k + 1\right)\left(\frac{2^{n-k}-1}{n-k}\right)^2\Bigg]
\end{aligned}
$$

- Theorem (B. and Reitzner, 1997):

  Let K be a plane convex body of unit area, and consider the family of all directed chords that divide K into a part of area s to the left and a part of area $1 - s$ to the right. The locus of the midpoints of these chords is a closed curve M_s. Let the orientation of M_s correspond to the chords rotating in the counter-clockwise direction. Put

$$K_{[s]} := 1 - \int_{z \in K \setminus M_s} w(z, M_s) \mathrm{d}z,$$

  where $w(z, M_s)$ is the winding number of M_s about the point z.

Then

$$p_{jk}(K) = \frac{4}{3}\, jk \int_0^1 s^{j-1}(1-s)^{k-1} K_{[s]} \mathrm{d}s.$$

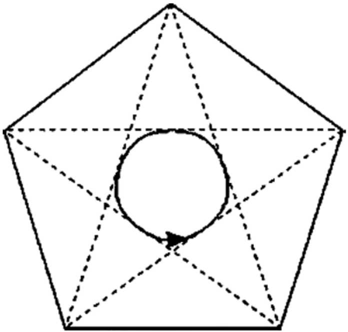
Especially, the expected number $\mathbb{E}N_n(K)$ of vertices of the convex hull of n points distributed independently and uniformly in K is given by

$$\mathbb{E}N_n(K) = n\, p_{n-1,1}(K) = \frac{4}{3} n(n-1) \int_0^1 s^{n-2} K_{[s]} \mathrm{d}s$$

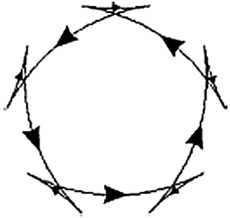
and the expected area $\mathbb{E}A_n(K)$ of the convex hull of n such points by

$$\mathbb{E}A_n(K) = 1 - p_{n,1}(K) = 1 - \frac{4}{3} n \int_0^1 s^{n-1} K_{[s]} \mathrm{d}s.$$

- M_s for the regular pentagon

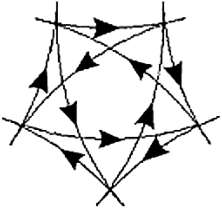


(a) $s = \frac{1}{2} - \frac{\sqrt{5}}{10} \approx 0.276393$
1 : 1

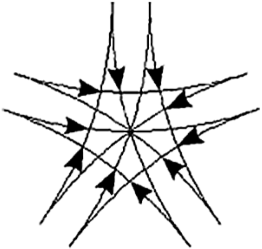


(b) $s = 0.4$
1 : 0.3

- M_s for the regular pentagon

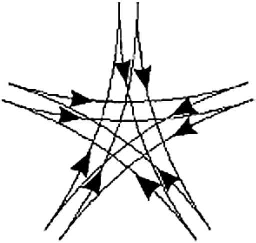


(c) $s = 0.46$
1 : 0.2

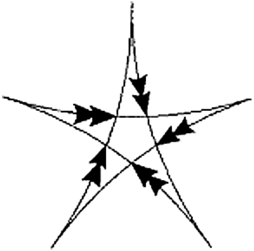


(d) $s = \frac{2}{5} + \frac{\sqrt{5}}{25} \approx 0.489443$
1 : 0.15

- M_s for the regular pentagon



(e) $s = 0.495$
1 : 0.15



(f) $s = 0.5$
1 : 0.15

- Theorem (B. and Reitzner, 1997):

  Let K be a convex polygon of unit area with vertices $v_1, \ldots, v_r$. Denote by s_{hi} the area of the convex hull of the vertices $v_h, v_{h+1}, \ldots, v_{i-1}, v_i$. Put

$$\frac{1}{f_{hi}} := \begin{cases} 0 & \text{if } \overline{v_h v_{h+1}} \text{ and } \overline{v_i v_{i+1}} \text{ are parallel,} \\ \frac{s_{hi} + s_{h+1,i+1} - s_{h,i+1} - s_{h+1,i}}{s_{hi} s_{h+1,i+1} - s_{h,i+1} s_{h+1,i}} & \text{else.} \end{cases}$$

Then

$$\begin{aligned}\mathbb{E}N_n(K) \;&= \frac{2r}{3}(\log n + C) + \frac{2}{3}\sum_{i=1}^{r}\log s_{i,i+2} + \frac{r}{3n} - \\ &- \frac{2r}{3}\sum_{m=1}^{\infty}\frac{B_{2m}}{2m}\frac{1}{n^{2m}} - \frac{2}{3}\sum_{m=1}^{\infty}\frac{n!(m-1)!}{(n+m)!}\sum_{i=1}^{r}\frac{1}{f_{i,i+2}^{m}} + \\ &+ O(\max_{i=1,\dots,r}(1-s_{i,i+2})^n),\end{aligned}$$

where the constants B_{2m} are the Bernoulli numbers.

- „Classical“ Hypergeometric Function:

$$
\begin{aligned}
{}_2F_1(\alpha_1,\alpha_2;\beta_1;x) = \\
&= 1 + \frac{\alpha_1\alpha_2}{\beta_1}\frac{x}{1!} + \frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)}{\beta_1(\beta_1+1)}\frac{x^2}{2!} + \cdots = \\
&= \sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k}{(\beta_1)_k}\frac{x^k}{k!},
\end{aligned}
$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

- Generalized Hypergeometric Function:

$$
{}_AF_B(\alpha_1, \alpha_2, \ldots, \alpha_A; \beta_1, \beta_2, \ldots, \beta_B; x) = \\
= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k \cdots (\alpha_A)_k}{(\beta_1)_k (\beta_2)_k \cdots (\beta_B)_k} \frac{x^k}{k!}.
$$

- Theorem (B. and Reitzner, 2001)

  The expected volume $\mathcal{V}(n)$ of the convex hull of n random points chosen independently and uniformly from a tetrahedron of volume one is given by:

$$
\begin{aligned}
\mathcal{V}(n) &= 1 - \frac{2}{n+1} - \frac{3(n-1)n}{4}\left[\frac{1}{(n+1)^3} + \sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}\frac{1}{(k+3)^3}\right] \\
&\quad - \frac{9(n-1)n}{2} \sum_{\substack{j_1+\ldots+j_5=n-2 \\ k_1+k_2+k_3=4 \\ j_1,\ldots,j_5,k_1,k_2,k_3\geq 0}} \binom{n-2}{j_1,\ldots,j_4}\binom{4}{k_1,k_2} 2^{k_2}\; 3^{\,j_2+j_3} \\
&\quad \times B(j_2+2j_3+3j_4+3j_5+k_2+2k_3+1, 3j_1+2j_2+j_3+2k_1+k_2+1) \\
&\quad \times B(n+1, j_5+k_3+1)\; B(2j_1+j_2+k_1+1, j_5+2) \\
&\quad \times {}_3F_2(j_5+k_3, n+1, 2j_1+j_2+k_1+1; j_5+k_3+n+2, 2j_1+j_2+j_5+k_1+3; 1) \\
&\quad + 6(n-1)n \sum_{\substack{j_1+\ldots+j_5=n-2 \\ l_1+l_2=2 \\ l_3+l_4=2 \\ j_1,\ldots,j_5,l_1,l_2,l_3,l_4\geq 0}} \binom{n-2}{j_1,\ldots,j_4}\binom{2}{l_1}\binom{2}{l_3} 3^{\,j_2+j_3} \\
&\quad \times B(j_2+2j_3+3j_4+3j_5+l_2+l_4+3, 3j_1+2j_2+j_3+l_1+l_3+3) \\
&\quad \times B(n+1, j_5+l_4+1)\; B(2j_1+j_2+l_1+1, j_5+3) \\
&\quad \times {}_3F_2(j_5+l_4+1, n+1, 2j_1+j_2+l_1+1; j_5+l_4+n+2, 2j_1+j_2+j_5+l_1+4; 1).
\end{aligned}
$$

- The first non-trivial values are:

$$\begin{aligned}
\mathcal{V}(4) &= \frac{13}{720} - \frac{\pi^2}{15015} && \approx 0.0173 \\
\mathcal{V}(5) &= \frac{13}{288} - \frac{\pi^2}{6006} && \approx 0.0434 \\
\mathcal{V}(6) &= \frac{127}{1680} - \frac{89\pi^2}{323323} && \approx 0.0728 \\
\mathcal{V}(7) &= \frac{307}{2880} - \frac{211\pi^2}{554268} && \approx 0.1028 \\
\mathcal{V}(8) &= \frac{41369}{302400} - \frac{22829\pi^2}{47805615} && \approx 0.1320 \\
\mathcal{V}(9) &= \frac{11129}{67200} - \frac{461\pi^2}{817190} && \approx 0.1600 \\
\mathcal{V}(10) &= \frac{641303}{332640} - \frac{3058061\pi^2}{4775249765} && \approx 0.1864 \\
\mathcal{V}(11) &= \frac{37723}{172800} - \frac{6445438\pi^2}{9116385915} && \approx 0.2113
\end{aligned}$$

- Identity of the values $\mathcal{V}(n)$:

For $m = 2, 3, \ldots$

$$\mathcal{V}(2m+1) = \sum_{k=1}^{m-1} (2^{2k} - 1)\frac{B_{2k}}{k}\binom{2m+1}{2k-1}\mathcal{V}(2m-2k+2),$$

where the constants B_{2k} are the Bernoulli numbers

$$B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}, \ldots.$$

- Thus the values are related by:

$$\begin{aligned}
\mathcal{V}(5) &= \frac{5}{2}\,\mathcal{V}(4),\\
\mathcal{V}(7) &= \frac{7}{2}\,\mathcal{V}(6) - \frac{35}{4}\,\mathcal{V}(4),\\
\mathcal{V}(9) &= \frac{9}{2}\,\mathcal{V}(8) - 21\,\mathcal{V}(6) + 63\,\mathcal{V}(4),\\
\mathcal{V}(11) &= \frac{11}{2}\,\mathcal{V}(10) - \frac{165}{4}\,\mathcal{V}(8) + 231\,\mathcal{V}(6) - \frac{2805}{4}\,\mathcal{V}(4).
\end{aligned}$$

- Bárány and B. (1993)

$$\mathbb{E}N_n(P) = \frac{F(P)}{(d+1)^{d-1}(d-1)!}\log^{d-1} n + O\left(\log^{d-2} n \log\log n\right)$$

- Groeneboom (1988)

$$\frac{N_n - \frac{2}{3} r \log n}{\sqrt{\frac{10}{27} r \log n}} \xrightarrow{D} \mathcal{N}(0,1)$$

- Cabo and Groeneboom (1994)

$$\frac{D_n - \frac{2}{3} r n^{-1} \log n}{\sqrt{\frac{100}{189} r n^{-1} \log n}} \xrightarrow{D} \mathcal{N}(0,1)$$

- Efron (1965)

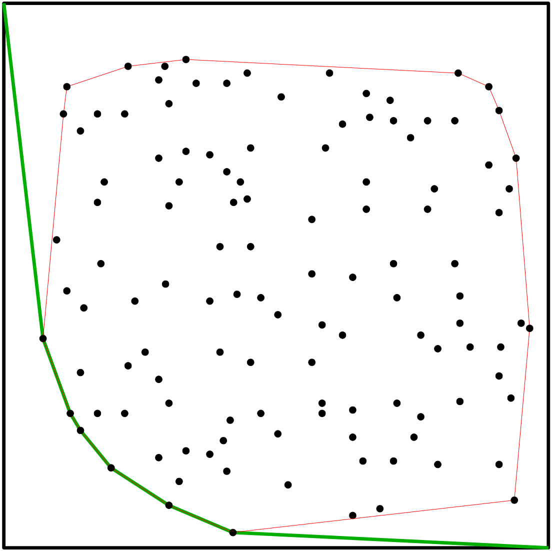
$$\frac{\mathbb{E}V_n}{\operatorname{vol} K} = 1 - \frac{\mathbb{E}N_{n+1}}{n+1}$$

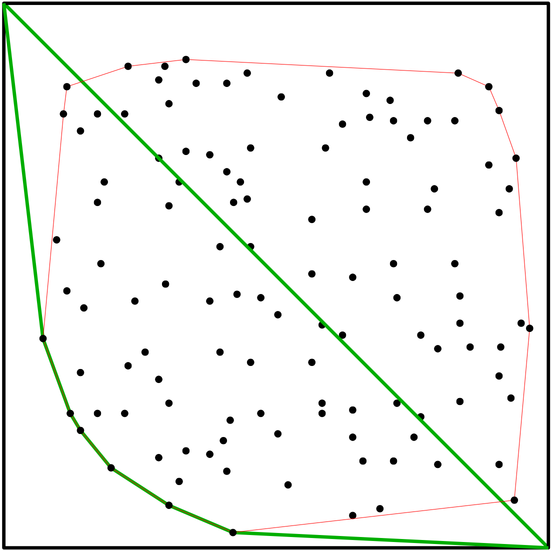
- B. (2000/2005)

$$\frac{\mathbb{E}V_n^k}{(\operatorname{vol} K)^k} = \mathbb{E}\prod_{i=1}^{k}\left(1 - \frac{N_{n+k}}{n+i}\right)$$

Thus the k^{th} moment of V_n can be expressed by the first k moments of N_{n+k}:

$$\begin{aligned}\frac{\mathbb{E}V_n^k}{(\operatorname{vol} K)^k} &= 1 - \left(\frac{1}{n+1} + \cdots + \frac{1}{n+k}\right)\mathbb{E}N_{n+k} \\ &\quad + \cdots + (-1)^k \frac{1}{(n+1)\cdots(n+k)}\mathbb{E}N_{n+k}^k.\end{aligned}$$





- Bárány, Rote, Steiger and Zhang (2000)

$$q_n^{(n)} = \frac{2^n}{n!(n+1)!}$$

- B. (2006)

$$\begin{aligned} q_k^{(n)} &= 2^k \sum \frac{1}{i_1(i_1+i_2)\cdots(i_1+i_2+\ldots+i_k)} \\ &\quad \times \frac{i_1 i_2 \cdots i_k}{(i_1+1)(i_1+i_2+1)\cdots(i_1+i_2+\ldots+i_k+1)} \end{aligned}$$

where $i_1, i_2, \ldots, i_k \in \mathbb{N}$ such that $i_1 + i_2 + \ldots + i_k = n$

Recurrence formulae:

- $q_n^{(n)} = \frac{2}{n(n+1)} q_{n-1}^{(n-1)}$
- $q_k^{(n)} = \frac{2}{n(n+1)} \sum_{j=k-1}^{n-1} (n-j) q_{k-1}^{(j)}$
- $\frac{n(n+1)}{2} q_k^{(n)} - (n-1)n q_k^{(n-1)} + \frac{(n-2)(n-1)}{2} q_k^{(n-2)} = q_{k-1}^{(n-1)}$

Hence,

$$\frac{n(n+1)}{2}\mathbb{E}N_n^m - (n^2 - n + 1)\mathbb{E}N_{n-1}^m + \frac{(n-2)(n-1)}{2}\mathbb{E}N_{n-2}^m$$

$$= \sum_{j=1}^{m} \binom{m}{j} \mathbb{E}N_{n-1}^{m-j}.$$

We study now the difference equation

$$\frac{n(n+1)}{2}f(n) - (n^2 - n + 1)f(n-1) + \frac{(n-2)(n-1)}{2}f(n-2) = g(n).$$

Expected value and variance:

$$\mathbb{E}N_n = \frac{2}{3}\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{3}$$

$$\mathbb{E}N_n^2 = \frac{4}{9}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2 + \frac{22}{27}\sum_{k=1}^{n}\frac{1}{k} + \frac{4}{9}\sum_{k=1}^{n}\frac{1}{k^2} - \frac{25}{27} + \frac{4}{9(n+1)}$$

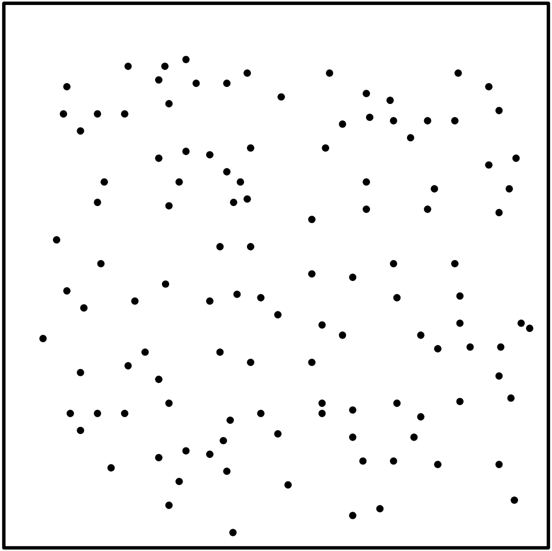
$$\operatorname{var} N_n = \frac{10}{27}\sum_{k=1}^{n}\frac{1}{k} + \frac{4}{9}\sum_{k=1}^{n}\frac{1}{k^2} - \frac{28}{27} + \frac{4}{9(n+1)}$$

Third Moment:

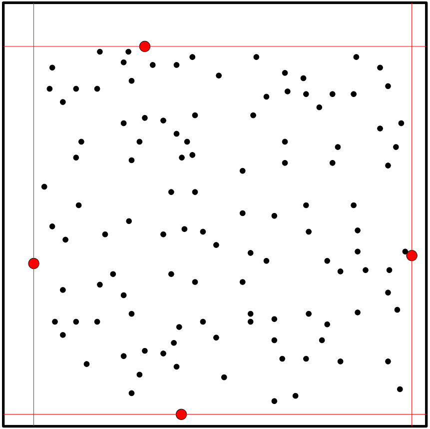
$$\begin{aligned}\mathbb{E}N_n^3 \;=\; & \frac{8}{27}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^3+\frac{32}{27}\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2+\left(\frac{8}{9}\sum_{k=1}^{n}\frac{1}{k^2}-\frac{106}{81}\right)\sum_{k=1}^{n}\frac{1}{k}\\ & -\frac{16}{9}\sum_{k_2=1}^{n}\frac{1}{k_2^2}\sum_{k_1=1}^{k_2}\frac{1}{k_1}+\frac{32}{27}\sum_{k=1}^{n}\frac{1}{k^2}+\frac{16}{27}\sum_{k=1}^{n}\frac{1}{k^3}+\frac{91}{81}\\ & -\frac{8}{9n}\sum_{k=1}^{n}\frac{1}{k}-\frac{16}{27(n+1)}\end{aligned}$$

Asymptotic behaviour of the m^{th} moment:

$$\mathbb{E}N_n^m=\left(\frac{2}{3}\log n\right)^m+O\left(\log^{m-1} n\right)$$

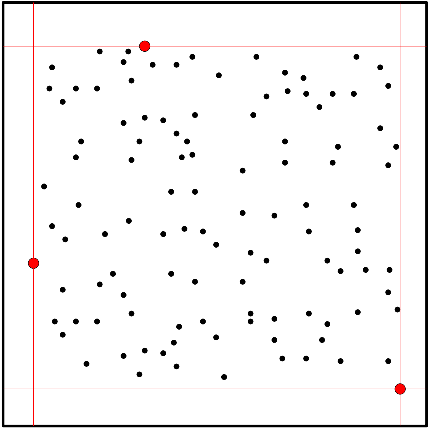


Position of the points: Type (a)



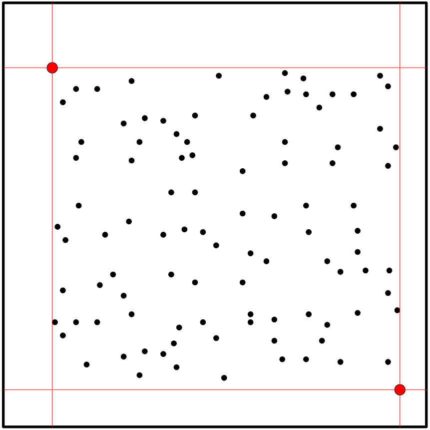
$$p_a^{(n)} = \frac{(n-2)(n-3)}{n(n-1)}$$

Position of the points: Type (b)

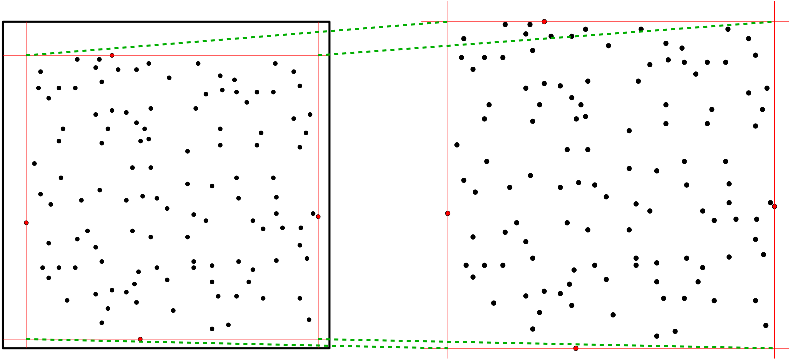


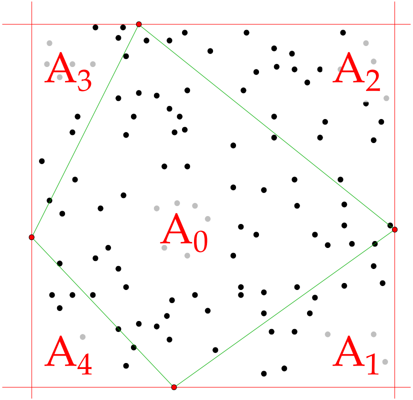
$$p_b^{(n)} = \frac{4(n-2)}{n(n-1)}$$

Position of the points: Type (c)

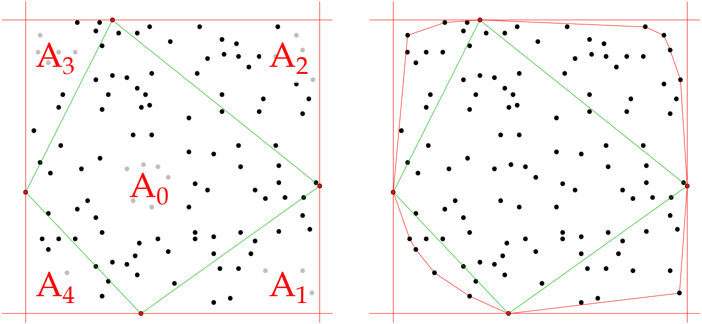


$$p_c^{(n)} = \frac{2}{n(n-1)}$$



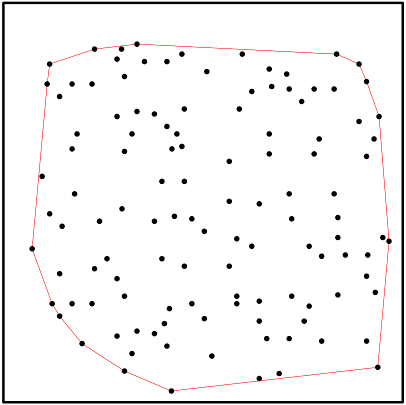


$$A_0 = 1 - \frac{1}{2}\left(x_1 + x_2 + x_3 + x_4 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_1\right)$$
$$A_1 = \frac{1}{2}\, x_1(1 - x_2)$$
$$A_2 = \frac{1}{2}\, x_2(1 - x_3)$$
$$A_3 = \frac{1}{2}\, x_3(1 - x_4)$$
$$A_4 = \frac{1}{2}\, x_4(1 - x_1)$$



$$P^{(n)}_{k_1,k_2,k_3,k_4}(x_1,x_2,x_3,x_4) = (n-4)! \sum_{\substack{r_0+\cdots+r_4=n-4 \\ r_0 \geq 0 \\ r_1 \geq k_1,\ldots,r_4 \geq k_4}} \frac{1}{r_0!} A_0^{r_0} \prod_{i=1}^{4} \frac{1}{r_i!} q_{k_i}^{(r_i)} A_i^{r_i}$$

Probability that the convex hull in the square has exactly k vertices:



$$p_k^{(n)}(\text{square}) = p_a^{(n)} p_{k|a}^{(n)} + p_b^{(n)} p_{k|b}^{(n)} + p_c^{(n)} p_{k|c}^{(n)}$$

Probability that the convex hull in the square is of a certain type and has exactly k vertices ($n = 7$):

k	3	4	5	6	7	Σ
a	0	$\frac{65}{1008}$	$\frac{155}{672}$	$\frac{475}{3024}$	$\frac{145}{6048}$	$\frac{10}{21}$
b	$\frac{137}{10080}$	$\frac{1811}{12600}$	$\frac{1046}{4725}$	$\frac{223}{2520}$	$\frac{1361}{151200}$	$\frac{10}{21}$
c	$\frac{1}{1008}$	$\frac{437}{25200}$	$\frac{1621}{75600}$	$\frac{109}{15120}$	$\frac{1}{1575}$	$\frac{1}{21}$
Σ	$\frac{7}{480}$	$\frac{203}{900}$	$\frac{3409}{7200}$	$\frac{91}{360}$	$\frac{121}{3600}$	1

Expected value and variance of $N_n(\text{triangle})$:

$$\mathbb{E} N_n(\text{triangle}) = 2 \sum_{k=1}^{n-1} \frac{1}{k}$$

$$\operatorname{var} N_n(\text{triangle}) = \frac{10}{9} \sum_{k=1}^{n-1} \frac{1}{k} - \frac{4}{3} \sum_{k=1}^{n-1} \frac{1}{k^2}$$

Recent important contributions are due to:

Imre Bárány
Pierre Calka
Xavier Goaoc
Matthias Reitzner
Tomasz Schreiber
...