"Raising expectations"

On the monotonicity of the expected size of a random polytope.

Xavier Goaoc (INRIA - LORIA)

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Sariel Har-Peled (UIUC), Guillaume Moroz (INRIA),
Raimund Seidel (Univ. des Saarlandes)

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Convex polyhedron, smooth convex body, weirder convex bodies...

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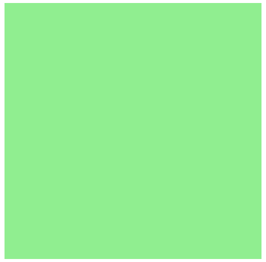

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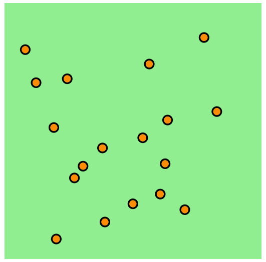

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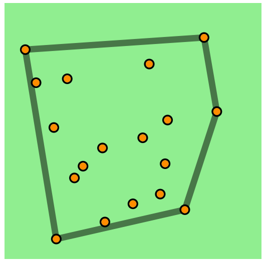

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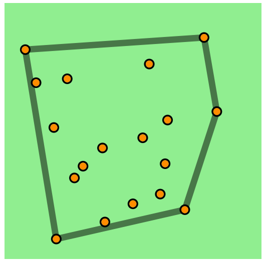

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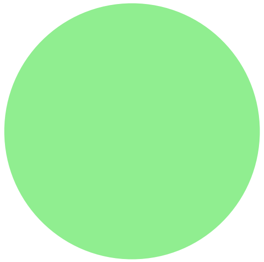

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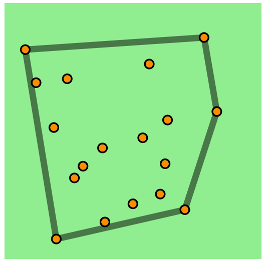

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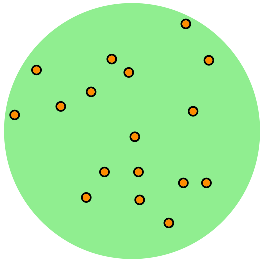

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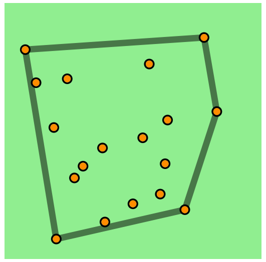

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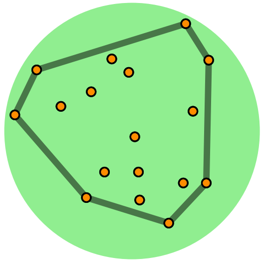

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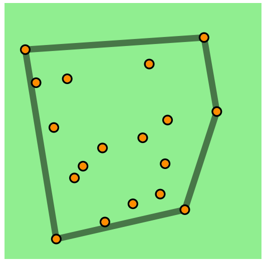

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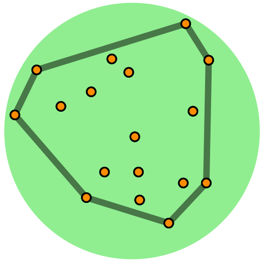

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Asymptotic behavior of the expected number of k-faces, the expected volume...

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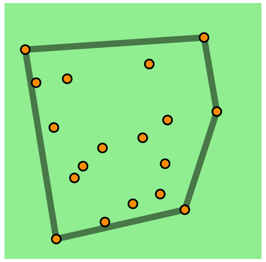

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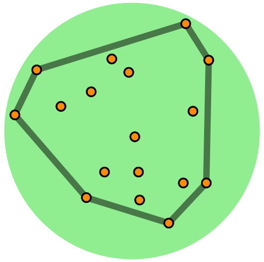

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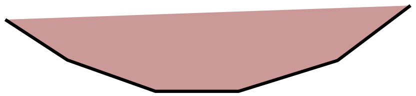
Yet amazingly "simple" questions remain open. Here's one:

Fix K and define $f_i(K_n)$ as the number of i-dimensional faces of K_n.

Is $n \mapsto E[f_0(K_n)]$ a monotone function?

Isn’t this obvious?



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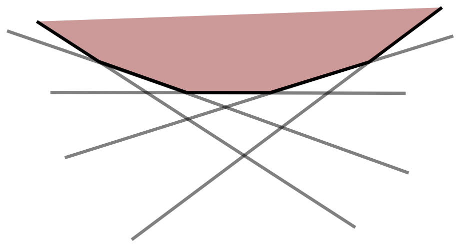

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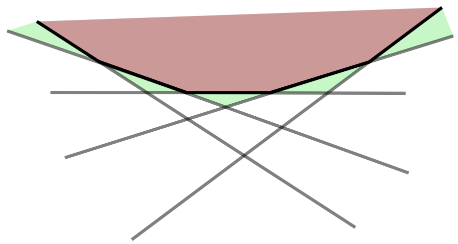

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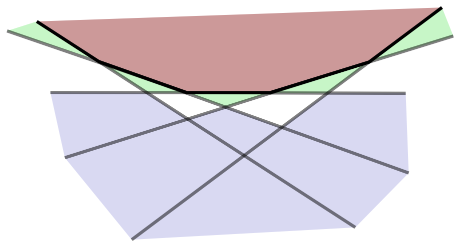

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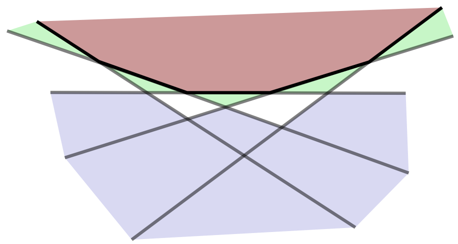

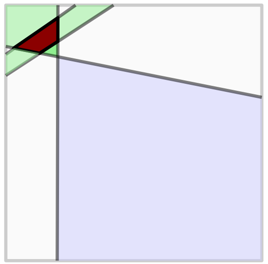

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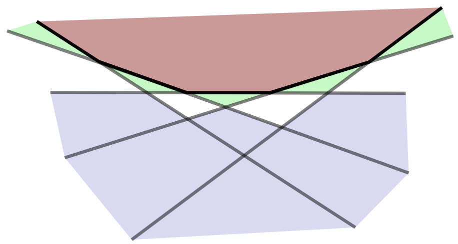

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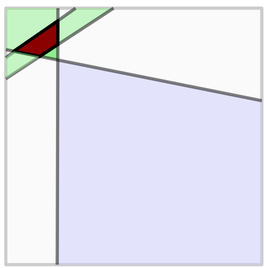

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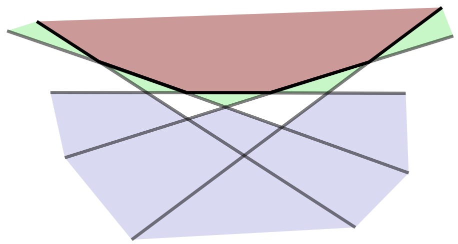

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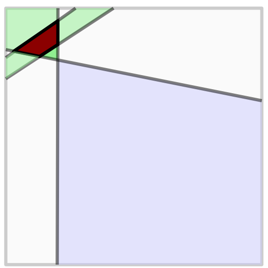

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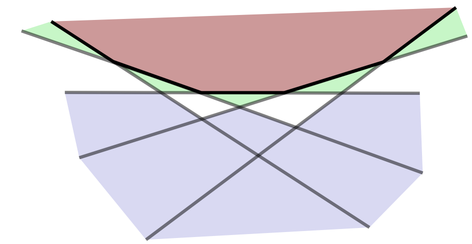

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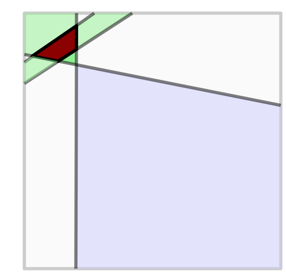

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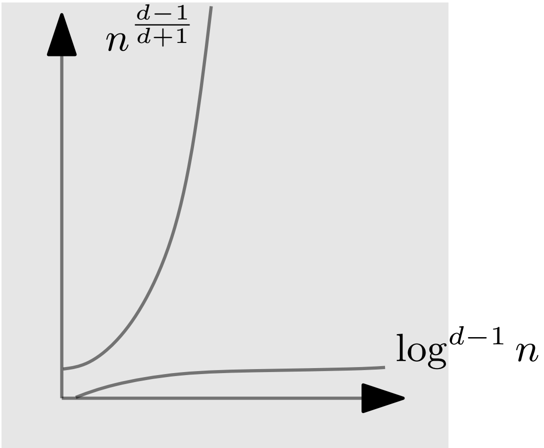

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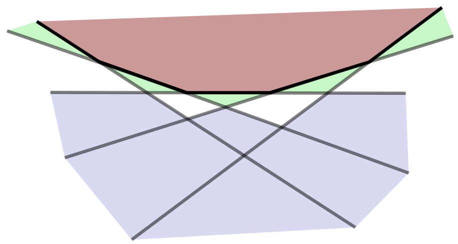

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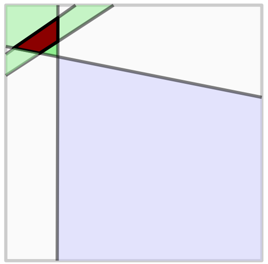

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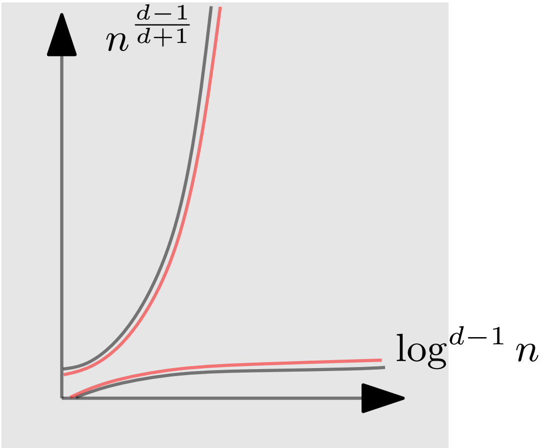

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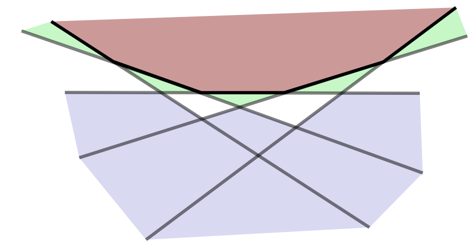

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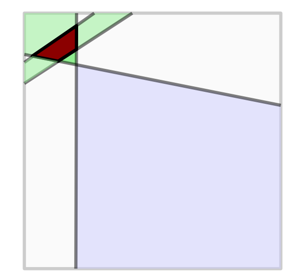

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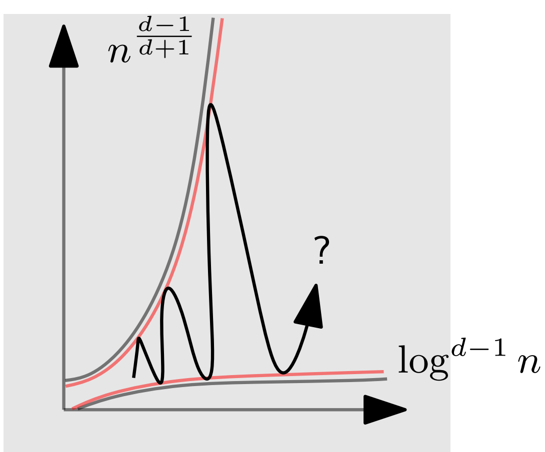

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There are also some algorithmic motivations:

I'd like to generate sets of points with particular properties.

Near-uniform distribution of combinatorial types.

Avoiding certain patterns.

A natural approach is to do it incrementally.

Monotonicity of f_0 is a basic "incremental" property.

What is known?

Exact formula for convex polygonal domains in the plane.

[Buchta-Reitzner 1997]

For any compact convex planar domain K the map $n \mapsto E[f_0(K_n)]$ is increasing.

[Reitzner 201?]

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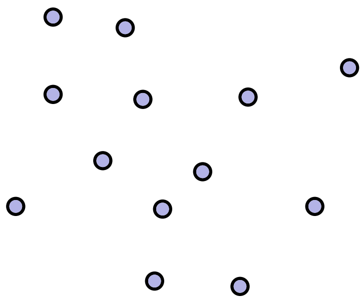
Based on a sampling argument ("Clarkson-Shor") introduced to analyze deterministic geometric configurations.

The Clarkson-Shor technique through an example

Let X be a finite generic set of points in $\mathbb{R}^2$.



A k-set of X is a pair (a, b) such that the open halfplane to the left of the line (ab), oriented from a to b, contains k points of X.

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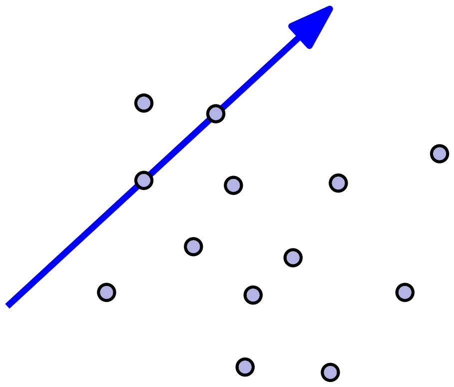

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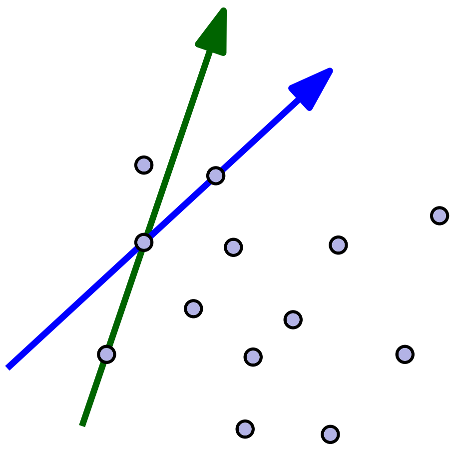

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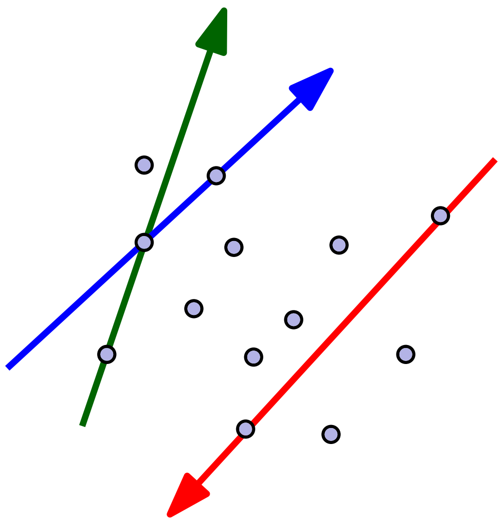

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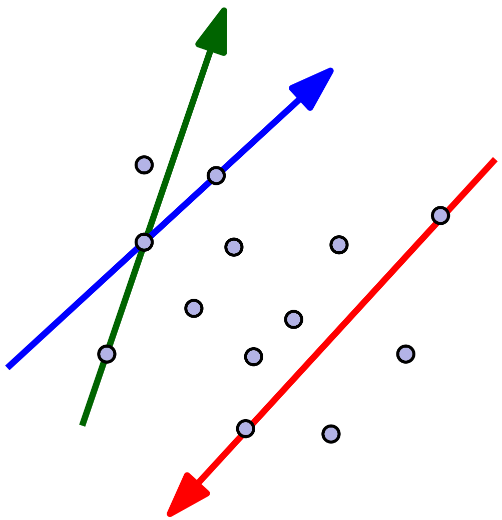

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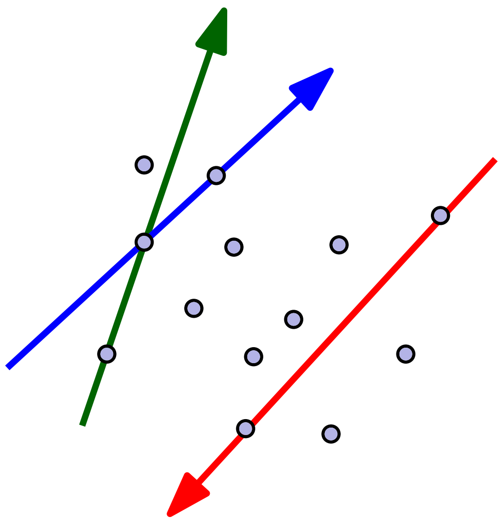

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Let R be a random sample of X.

Flip a (biased) coin for each point independently to decide whether to keep it or not.

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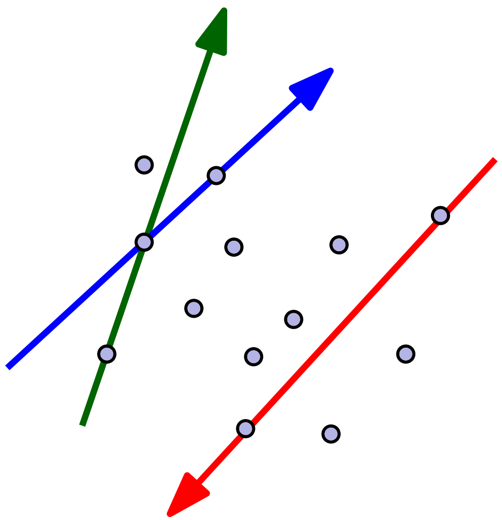

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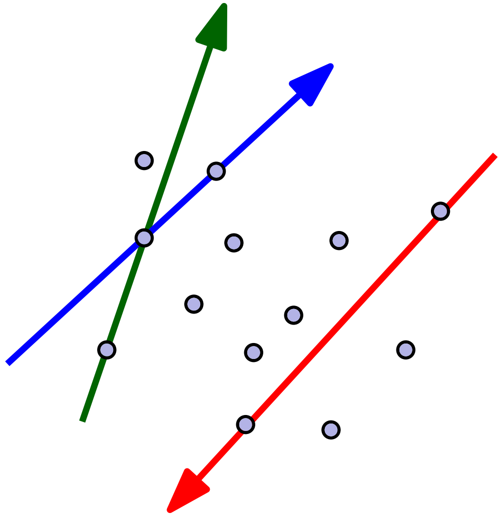

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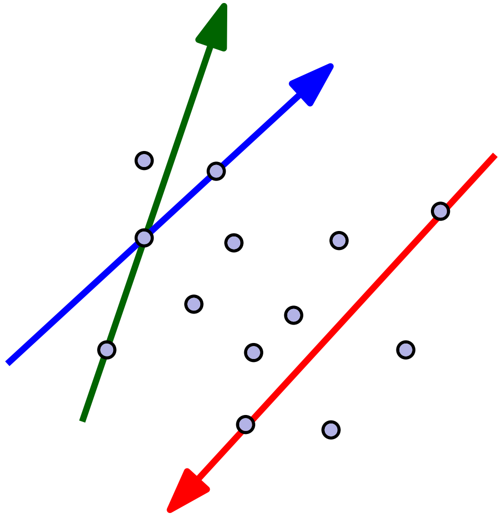

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$$\wedge|$$

$$E[\#R] = pn$$

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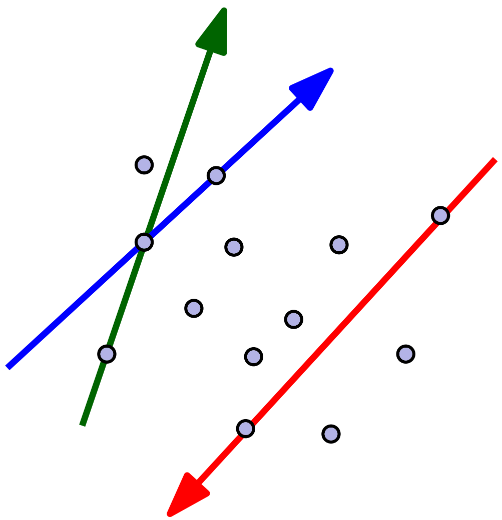

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$s_i(X) =$ number of i-sets of X.

Let R be a random sample of X.

Flip a (biased) coin for each point independently to decide whether to keep it or not.

$$E[s_0(R)] = \sum_{i=0}^{n-2} s_i(X) p^2 (1-p)^i \geq \sum_{i=0}^{k} s_i(X) p^2 (1-p)^i \geq p^2 (1-p)^k \sum_{i=0}^{k} s_i(X)$$

$$\wedge|$$

$$E[\#R] = pn$$

$\Rightarrow$ for any choice of p we have $\sum_{i=0}^{k} s_i(X) \leq \frac{n}{p(1-p)^k}$.

Setting $p = \frac{1}{k}$ yields that $\sum_{i=0}^{k} s_i(X) = O(nk)$ for any n-points set X.

Expected size of the convex hull

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Step 1. $s_0(n) - s_0(n-1) \geq \frac{ds_0(n) - s_1(n)}{n}$

Step 2. $s_0(r) \geq x^d s_0(n) + x^d(1-x)s_1(n)$ where $1 \leq r \leq n$ and $x = \frac{r-d}{n-d}$.

Step 3. Use the asymptotic behavior of $f_{d-1}(n) = s_0(n)$ to express $s_0(r)$ in terms of $s_0(n)$.

Step 1. $s_0(n) - s_0(n-1) \geq \frac{ds_0(n) - s_1(n)}{n}$

Let Z be a set of n points chosen independently from $\mathcal{P}$.

Pick a point $q \in Z$ and remove it.

$$s_0(S) = s_0(Z \setminus \{q\}) + \sum_{F \text{ facet of } CH(Z)} \mathbb{1}_{q \in F} - \#1-\text{sets cutting off } q$$

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Since every facet of $CH(Z)$ has at least d vertices (and $f_{d-1} = s_0$):

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Taking the expectation over the choice of Z:

$$ns_0(n) \geq ns_0(n-1) + ds_0(n) - s_1(Z)$$

Step 2. $s_0(r) \geq x^d s_0(n) + x^d(1-x)s_1(n)$ **where** $1 \leq r \leq n$ **and** $x = \frac{r-d}{n-d}$**.**

Let Z be a set of n points chosen independently from $\mathcal{P}$.

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Count how many of the 0- and 1-sets of Z became 0-sets of R:

$$s_0(R) \geq s_0(Z)\frac{\binom{n-d}{r-d}}{\binom{n}{r}} + s_1(Z)\frac{\binom{n-d-1}{r-d}}{\binom{n}{r}}$$

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Putting $x = \frac{r-d}{n-d}$ we have

$$\frac{\binom{n-d}{r-d}}{\binom{n}{r}} \geq x^d \quad \text{and} \quad \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} \geq x^d(1-x).$$

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Altogether, $\quad s_0(r) \geq x^d s_0(n) + x^d(1-x)s_1(n).$

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From Step 1 we have $f_{d-1}(n) > f_{d-1}(n-1) \Leftrightarrow ds_0(n) > s_1(n)$

From Step 2 we have $s_0(r) \geq x^d s_0(n) + x^d(1-x)s_1(n)$ where $x = \frac{r-d}{n-d}$.

We now find some value $r(n)$ such that for all n large enough

$$s_0(r) < ((d+1)x^d - dx^{d+1})s_0(n)$$

and thus for all n large enough $f_{d-1}(n) > f_{d-1}(n-1)$

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For any $\epsilon > 0$ there exists N_ϵ such that

for all $n \geq N_\epsilon$ we have $(1-\epsilon)An^c \leq f_{d-1}(n) \leq (1+\epsilon)An^c$

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Show that for n large enough there exists $r \geq N_\epsilon$ such that

$$\frac{1+\epsilon}{1-\epsilon}\left(\frac{r}{n}\right)^c < ((d+1)x^d - dx^{d+1}) \text{ where } x = \frac{r-d}{n-d}.$$

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and thus for all n large enough $f_{d-1}(n) > f_{d-1}(n-1)$

A sufficient condition is that $\lim_{n\to\infty} \frac{f_{d-1}(n)}{An^c} = 1$ $\longleftarrow$ This is the only assumption we make on the distribution $\mathcal{P}$.

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From facets to other faces

So, when K is smooth $E[f_{d-1}(K_n)]$ is asymptotically increasing.

When K is polyhedral this technique cannot say much...

Since K_n is a. s. simplicial, $E[f_{d-2}(K_n)] = \frac{d}{2}E[f_{d-1}(K_n)]$ is also asymptotically monotone.

Every $(d-1)$-face is incident to d $(d-2)$-faces.

For $d = 3$ we have $f_0(K_n) = 2 - f_2(K_n) + f_1(K_n) = 2 + \frac{1}{2}f_2(K_n)$ by Euler's relation and $E[f_0(K_n)]$ is also asymptotically monotone.

To summarize

An interesting property.

Any sufficiently fast asymptotic growth of $E[f_{d-1}]$ *cannot hide oscillations.*

A simple yet efficient probabilistic technique to analyze geometric configurations.

Introduce a notion of "levels".

Sample the configuration and analyze how the levels "propagate".

Optimize the sampling rate.

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