A turning-band method for the simulation of anisotropic fractional Brownian fields

joint work with L. Moisan and F. Richard

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Outlines

Fractal analysis of medical images

Stochastic modeling

Simulation of 2d anisotropic fractional Brownian field

Fractal analysis of medical images

ROI medical images = textures

Goal : use texture analysis to extract diagnostically meaningful information

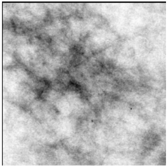
Method : fractal analysis to characterize texture via statistical self-similarity or scale invariance through a fractal index $H \in (0,1)$

Numerous methods and studies !

[Lopes and Betrouni, 2009]

Examples : Mammograms

dense breast tissue



fatty breast tissue

- Validation of self-similarity using power spectrum method [Heine et al, 2002]

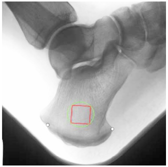
$$H \in [0.33, 0.42].$$

- Discrimination of dense and fatty breast tissue using WTMM method [Kestener et al, 2001]

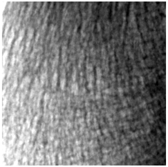
$H \in [0.55, 0.75]$ (dense breast tissue) $H \in [0.2, 0.35]$ (fatty breast tissue)

Examples : Trabecular bone microarchitecture

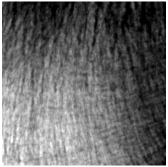
Dataset of 211 high-resolution digital X-ray images of calcaneum (a heel bone) with standardized acquisition procedure [Lespessailles et al., 2007] :



ROI location



control case



osteoporotic case

- Validation of self-similarity using variogram and power spectrum methods on calcaneous bone [Benhamou et al, 94], on cancellous bone [Caldwell et al, 94]
- Discrimination of osteoporotic cases [Benhamou et al, 2001]

$H_{mean} = 0.679 \pm 0.053$ (osteoporotic) $\quad H_{mean} = 0.696 \pm 0.030$ (control)

Fractional Brownian motion

For $H \in (0,1)$, the standard fractional Brownian motion [Kolmogorov, 1940], [Mandelbrot and Van Ness, 1968] $B_H = \{B_H(t)\,;\, t \in \mathbb{R}\}$ is a centered Gaussian process with stationary increments such that

$$\mathrm{Cov}\left(B_H(t), B_H(s)\right) = \frac{1}{2}\left(v_H(t) + v_H(s) - v_H(t-s)\right),$$

with $\forall t, s \in \mathbb{R}$, $v_H(t) = \mathbb{E}\left(\left(B_H(s+t) - B_H(s)\right)^2\right) = |t|^{2H}$.

Main Properties :

- H self-similarity : $\forall \lambda > 0$, $B_H(\lambda .) \overset{fdd}{=} \lambda^H B_H(.)$.
- H a.s. critical Hölder exponent :

$$|\widetilde{B_H}(t) - \widetilde{B_H}(s)| \leq C|t-s|^H |\log(|t-s|)|^{1/2} \text{ a.s.}$$

- H a.s. fractal dimension :

$$\dim_{\mathcal{H}}\left(\{(t, \widetilde{B_H}(t)), t \in [0,1]\}\right) = 2 - H \text{ a.s.}$$

Generalizations in dimension d

Spectral representation : for $\gamma(H) = \pi / H\Gamma(2H)\sin(H\pi)$,

$$\forall t \in \mathbb{R}, \qquad \underbrace{v_H(t)}_{\text{variogramme}} \qquad = \int_{\mathbb{R}} \left| e^{it\zeta} - 1 \right|^2 \underbrace{\gamma(H)^{-1} |\zeta|^{-2H-1}}_{\text{spectral density}} d\zeta.$$

Bochner Theorem : when μ is a Levy measure on $\mathbb{R}^d$ and

$$\forall x \in \mathbb{R}^d, \; v(x) = \int_{\mathbb{R}^d} \left| e^{ix\cdot\xi} - 1 \right|^2 d\mu(\xi),$$

$(x, y) \mapsto \frac{1}{2}(v(x) + v(y) - v(x - y))$ is the covariance function of a centered Gaussian random field with stationary increments.

Anisotropic fractional Brownian field

[Bonami, Estrade, 2003] for $c : S^{d-1} \to \mathbb{R}^+$ even in $L^1(S^{d-1})$ and $h : S^{d-1} \to (0,1)$ even, the afBf X is defined with

$$d\mu_X(\xi) = c\left(\frac{\xi}{\|\xi\|}\right)\|\xi\|^{-2h\left(\frac{\xi}{\|\xi\|}\right)-d}d\xi.$$

Main Properties : Let $H = \underset{\theta\in S^{d-1};c(\theta)>0}{\operatorname{essinf}}(h(\theta))$.

- H self-similarity iff $h = H$ a.e. : $\forall\lambda > 0,\ X(\lambda.) \overset{fdd}{=} \lambda^H X(.)$.
- H a.s. critical Hölder exponent :

$$|\widetilde{X}(x) - \widetilde{X}(y)| \leq C\|x-y\|^H|\log(\|x-y\|)|^{1/2} \text{ a.s.}$$

- H a.s. fractal dimension :

$$\dim_{\mathcal{H}}\left(\{(x,\widetilde{X}(x)), x \in [0,1]^d\}\right) = d+1-H \text{ a.s.}$$

- Isotropy if $h = H$ and $c = \underset{S^{d-1}}{\operatorname{essinf}}(c)$ a.e. (fBf) :

  $\forall R$ rotation, $X(R.) \overset{fdd}{=} X(.)$.

Simulation of a Gaussian vector

Let Y centered Gaussian vector of size n and covariance $C_Y \in \mathcal{M}_n(\mathbb{R})$ then

$$Y \stackrel{d}{=} R_Y \varepsilon_n \text{ with } C_Y = R_Y R_Y^t \text{ and } \varepsilon_n \sim \mathcal{N}(0, I_n).$$

Choleski method : to find R_Y cost $O(n^3)$.
Circulant matrix : when $C_Y = circ(c)$ with $c = (c_0\, c_1\, \dots\, c_{n-1})$ ie

$$C_Y = \begin{pmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{pmatrix}$$

then $C_Y = \frac{1}{n} F_n^* \mathrm{diag}(F_n c) F_n$ with F_n the matrix of discrete Fourier transform. Let $R_n = \frac{1}{\sqrt{n}} F_n^* \mathrm{diag}(F_n c)^{1/2} \in \mathcal{M}_n(\mathbb{C})$ then

$$Y \stackrel{d}{=} \Re(R_n(\varepsilon_n^1 + i\varepsilon_n^2)) \text{ with } \varepsilon_n^1, \varepsilon_n^2 \text{ iid } \mathcal{N}(0, I_n).$$

Cost $O(n \log(n))$ for $n = 2^p$.

Application to a stationary Gaussian process

Assume that $\mathrm{Cov}(Y_{k+l}, Y_l) = r_k$ s.t. $C_Y = \begin{pmatrix} r_0 & r_1 & \dots & r_n \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & r_0 \end{pmatrix}$

Embedding in a circulant matrix $S = circ(s)$ with

$$s = (r_0\, r_1\, \dots\, r_n\, [\dots]\, r_{n-1}\, \dots\, r_1) \text{ of size } M \geq 2n$$

such that

$$S = \begin{pmatrix} C_Y & S_1 \\ * & S_2 \end{pmatrix} \text{ satisfies } S = S^t.$$

If $F_M s \geq 0$ then S covariance and $Y \overset{d}{=} (Z_0, \dots, Z_n)$ for $Z \sim \mathcal{N}(0, S)$.

Remark : this is often difficult to find s satisfying this condition.

Fast and exact synthesis of 1d fBm

Let $H \in (0,1)$ and B_H a fBm. The fractional gaussian noise is defined as $Y_k = B_H(k+1) - B_H(k)$ so that

$$r_k = \mathrm{Cov}(Y_{k+l}, Y_l) = \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right).$$

Theorem [Perrin et al, 2002] : $\forall n \geq 1$, $S = circ(r_0\, r_1\, \ldots\, r_n\, r_{n-1}\, \ldots\, r_1)$ is a covariance matrix.

Since $B_H(0) = 0$ a.s., $B_H(k) = \sum_{l<k} Y_l$ for $k \geq 1$ and

- by stationarity of the increments

$$\left(B_H(k)\right)_{-m\leq k\leq n-m} \stackrel{d}{=} \left(\sum_{l<k+m} Y_l - \sum_{l<m} Y_l\right)_{-m\leq k\leq n-m}$$

- by self-similarity

$$\left(B_H(\lambda k)\right)_{0\leq k\leq n} \stackrel{d}{=} \lambda^H \left(B_H(k)\right)_{0\leq k\leq n}.$$

Extension to 2d Gaussian fields

- When stationarity and $\mathrm{Cov}(Y_{k_1+l_1,k_2+l_2}, Y_{l_1,l_2}) = r_{k_1,k_2}$ use a block Toeplitz covariance matrix with Toeplitz block and embed with a block circulant matrix [Chan, Wood, 1994, Dietrich, Newsam, 1997]
- When only stationarity increments simulate the increments but the initial conditions are correlated [Kaplan, Kuo, 1996]
- For the fBf approximate by a stationary field with compactly suported covariance function for which the circulant embedding matrix algorithm is running [Stein, 2002, Gneiting et al, 2006]
- Conditional simulation procedure when conditional covariances are known [Emery, Lantuejoul, 2006, Brouste et al, 2007]

Turning band method [Matheron, 1973]

When Y is a centered stationary process with covariance $C_Y(t,s) = r_Y(t-s)$ and $U \sim \mathcal{U}(S^1)$ define the field

$$Z(x) = Y(x \cdot U) \text{ for } x \in \mathbb{R}^2$$

such that with $u(\theta) = (\cos(\theta), \sin(\theta))$,

$$r(x) = \mathrm{Cov}(Z(x+y), Z(y)) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} r_Y(x \cdot u(\theta)) d\theta.$$

Then Z is a centered stationary isotropic field (not Gaussian). Defining for $\theta_1, \ldots, \theta_K \in [-\pi/2, \pi/2]$ and $\lambda_1, \ldots, \lambda_K \in \mathbb{R}^+$

$$Z_K(x) = \sum_{i=1}^{K} \sqrt{\lambda_i} Y^{(i)}(x \cdot u(\theta_i)),$$

with $Y^{(1)}, \ldots, Y^{(K)}$ independent realizations of Y the field Z_K is a centered stationary field with

$$r_K(x) = \sum_{i=1}^{K} \lambda_i r_Y(x \cdot u(\theta_i)).$$

Variogram of anisotropic fractional Brownian field

Let X be an afBf and recall that $\gamma(H) = \pi / H\Gamma(2H)\sin(H\pi)$.

$$\begin{aligned} v(x) &= \int_{\mathbb{R}^2} \left| e^{ix\cdot\xi} - 1 \right|^2 c\left(\xi/\|\xi\|\right)\|\xi\|^{-2h(\xi/\|\xi\|)-2} d\xi \\ &= \int_0^{2\pi}\int_0^{+\infty} \left| e^{ir(x\cdot u(\theta))} - 1 \right|^2 c(\theta) r^{-2h(\theta)-1} dr d\theta \\ &= \int_{-\pi/2}^{\pi/2} \gamma(h(\theta)) c(\theta) |x \cdot u(\theta)|^{2h(\theta)} d\theta. \end{aligned}$$

Let $(B^{(i)}_{h(\theta_i)})_{1\leq i\leq K}$ independent realizations of 1d fBm and define

$$X_K(x) = \sum_{i=1}^{K} \sqrt{\lambda_i \gamma(h(\theta_i)) c(\theta_i)} B^{(i)}_{h(\theta_i)}(x \cdot u(\theta_i)).$$

Then, $$\begin{aligned} d_{Kol}(X_K(x), X(x)) &= \sup_{t\in\mathbb{R}} |\mathbb{P}(X_K(x) \leq t) - \mathbb{P}(X(x) \leq t)| \\ &\leq 2|v_K(x) - v(x)|/v(x), \end{aligned}$$

with $v_K(x) = \sum_{i=1}^{K} \lambda_i \gamma(h(\theta_i)) c(\theta_i) |x \cdot u(\theta_i)|^{2h(\theta_i)}$.

Choice of lines

To simulate $B^{(i)}_{h(\theta_i)}(x \cdot u(\theta_i))$ for $x \in [0,1]^2 \cap n^{-1}\mathbb{Z}^2$ one has to simulate

$$B^{(i)}_{h(\theta_i)}\left(\frac{k}{n}\cos(\theta_i) + \frac{l}{n}\sin(\theta_i)\right) \text{ for } 0 \leq k, l \leq n.$$

When $\cos(\theta_i) \neq 0$ choose θ_i such that $\tan(\theta_i) = \frac{p_i}{q_i}$ with $p_i \in \mathbb{Z}$ and $q_i \in \mathbb{N}$ such that

$$\left(B^{(i)}_{h(\theta_i)}\left(\frac{k}{n}\cos(\theta_i) + \frac{l}{n}\sin(\theta_i)\right)\right)_{k,l} \overset{fdd}{=} \left(\frac{\cos(\theta_i)}{nq_i}\right)^{h(\theta_i)} \left(B^{(i)}_{h(\theta_i)}(kq_i + lp_i)\right)_{k,l}.$$

Cost : $O(n(|p_i| + q_i)\log(n(|p_i| + q_i)))$

➥ Choice of (θ_i) that minimizes the cost via dynamic programming.

Choice of weights

- **Rectangle rule :** h, c piecewize $\mathcal{C}^1$, $\lambda_i = \frac{\theta_{i+1} - \theta_i}{2}$ then

$$d_{Kol}(X_K(x), X(x)) = O(K^{-\min(2H,1)})$$

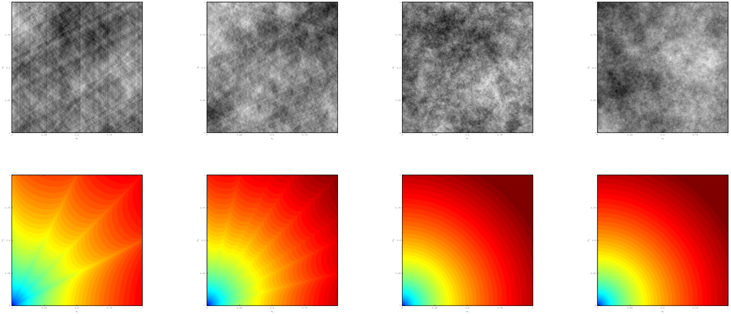
- **Trapezoidal rule :** h, c piecewize $\mathcal{C}^2$, $\lambda_i = \frac{\theta_{i+1} - \theta_{i-1}}{2}$ then

$$d_{Kol}(X_K(x), X(x)) = O(K^{-\min(2H,1)-1}\log(K))$$

Moreover, these hold uniformly on x in a compact set when h is constant.

For comparison in general the TBMs lead to a non-Gaussian field for which the Kolmogorov distance is $O(K^{-1/2})$ by Berry-Esseen Theorem (see [Emery, Lantuejoul, 2008] for FBF)

Number of lines



$K = 9$ $K = 17$ $K = 53$ $K = 113$

Realizations of X_K for X fBf with $h = H = 0.2$ and $c = 1$.

Elementary afBf

Let $\alpha \in (0, \pi/2]$, $c = \mathbf{1}_{(-\alpha,\alpha)}$ π-periodic and $h = H \in (0,1)$. Let $X_{\boldsymbol{H},\alpha}$ be the corresponding afBf variogram. Then, for $\gamma(H) = \pi / H\Gamma(2H)\sin(H\pi)$

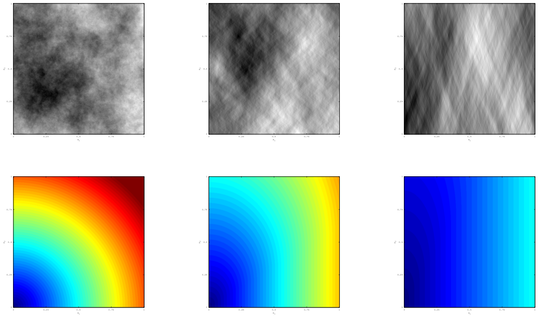
$$\forall x \in \mathbb{R}^2,\ \ v_{\boldsymbol{H},\alpha}(x) = 2^{2H}\gamma(H) C_H(arg(x)) \|x\|^{2H},$$

where C is a π periodic function defined on $(-\pi/2, \pi/2]$ by

$$C_H(\theta) = \begin{cases} \beta_{\boldsymbol{H}}\left(\frac{1-\sin(\alpha-\theta)}{2}\right) + \beta_{\boldsymbol{H}}\left(\frac{1+\sin(\alpha+\theta)}{2}\right) & \text{if} \quad -\alpha \leq \theta + \frac{\pi}{2} \leq \alpha \\ \beta_{\boldsymbol{H}}\left(\frac{1+\sin(\alpha-\theta)}{2}\right) + \beta_{\boldsymbol{H}}\left(\frac{1-\sin(\alpha+\theta)}{2}\right) & \text{if} \quad \alpha \leq \theta - \frac{\pi}{2} \leq \alpha \\ \left|\beta_{\boldsymbol{H}}\left(\frac{1-\sin(\alpha-\theta)}{2}\right) - \beta_{\boldsymbol{H}}\left(\frac{1+\sin(\alpha+\theta)}{2}\right)\right| & \text{otherwize} \end{cases}$$

with $\beta_{\boldsymbol{H}}(t) = \int_0^t u^{H-1/2}(1-u)^{H-1/2} du$ is a Beta incomplete function.

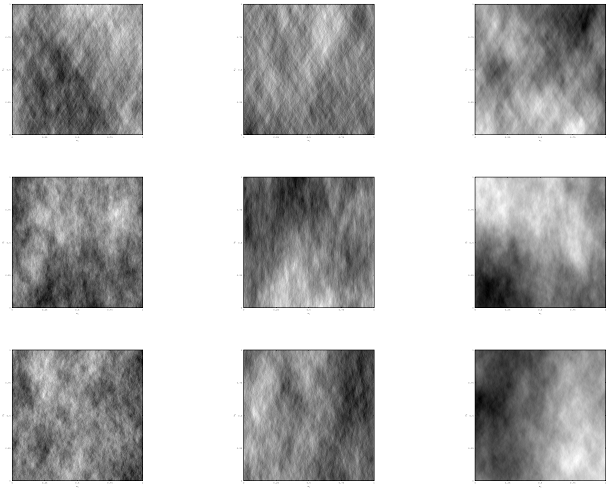
Elementary afBf



$\alpha = \pi/2$ $\alpha = \pi/3$ $\alpha = \pi/6$

Realizations of approximations of $X_{\boldsymbol{H},\alpha}$ for $K = 5900$ and $H = 0.5$.

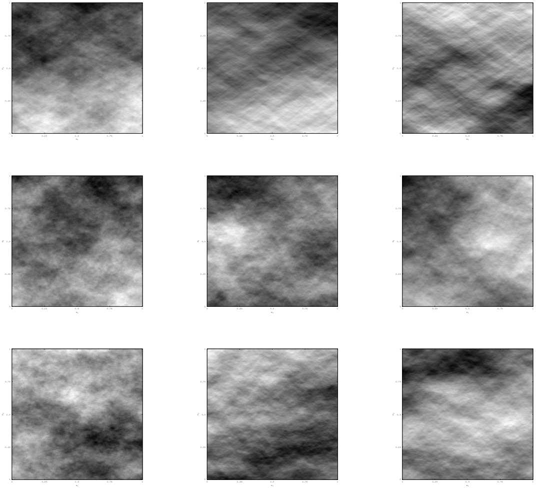
Other realizations



$H_1 = 0.2,\ H_2 = 0.5$ $\quad$ $H_1 = 0.2,\ H_2 = 0.8$ $\quad$ $H_1 = 0.5,\ H_2 = 0.8$

Realizations of X_K for $K = 517$, $c = 1$ with $h(0) = H_1$ and $h(\pm\pi/2) = H_2$, elementary (top), linear (middle), smooth (bottom)

Other realizations



$c_2 = 5$ $\quad$ $c_2 = 100$ $\quad$ $c_2 = 1000$

Realizations of X_K for $K = 517$, $h = H = 0.5$ with $c(0) = c_1$ and $c(\pm\pi/2) = c_2$, elementary (top), linear (middle), smooth (bottom)